TA Notes - Week 1
Math Review

Conor Foley[*]

Econ 102 (Geerolf) Fall 2018

Contents

1 Overview (How to Use this Note) 2

2 Appendix 2 Propositions 2
2.1 Geometric Series 2
2.2 Polynomial Approximations 2
2.3 Growth Rates and Growth Factors 3

3 Compound Growth 4

4 Exponents and Logarithms 4
4.1 Exponent and Logarithm Rules 5
4.2 Expressing any number using any base 5
4.3 Application - Compound Growth 6
4.4 Natural Logs and Percent Change 7

5 Functions 9
5.1 Linear Function - Basics 9
5.2 Linear Equations with Logs 10
5.3 Warning on Interpretation 11

6 Appendix A - Taylor Series and Approximation for $\ln(1+x)$ 11

7 Appendix B - Proof of Appendix 2 Approximations 12
7.1 Proof using Expansion 12
7.2 Proof using Log-Approximation 13
7.3 Proof using Taylor Approximation 14

8 Appendix C - Proof of Geometric Series Formula 15

[*]For errors or corrections, please email me at conor.teaches.econ@gmail.com.

1 Overview (How to Use this Note)

This note goes over some of the main math tools used in this class. Mastery of these math tools is not necessary on their own. The goal is to prepare you for how they are used in the main economics topics.

The discussion here is meant to expand on what is given in Appendix 2 and Appendix 3 of the Blanchard textbook, providing additional detail, explaining the connections between some of these concepts, and giving some background on why we choose to use these math tools.

The discussion of geometric series, and logs and growth, will be helpful mathematical tools. They are presented here in a general way, so that this note can serve as a reference both for this class and if you encounter these topics again in the future.

Proofs are given at the end of the note, but are completely optional. They are included for those with an interest in mathematical details or if you benefit in understanding/remembering a result by seeing where it comes from.

2 Appendix 2 Propositions

2.1 Geometric Series

Geometric Series:

(1) Finite Sum: $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

(2) Limit: If $|r| < 1$ then $\lim_{n\to\infty} \sum_{i=0}^{n} r^i = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$

- Comment: Note that if $|r| \geq 1$ then the limit formula fails because the sum does not converge as we add more terms.

2.2 Polynomial Approximations

Polynomial Approximations: Note that all of these approximations work well only when x and y are close to zero.

(3) $(1+x)(1+y) \approx 1 + x + y$

(4) $(1+x)^2 \approx 1 + 2x$

(5) $(1+x)^n \approx 1 + nx$

(6) $\frac{1+x}{1+y} \approx 1 + x - y$

2.3 Growth Rates and Growth Factors

Growth Rate of a Product: Suppose we have a variable z which is defined as the product (or division) of two other variables x and y. In addition, we can define the growth rate (in decimal terms) of these variables as g_{var} where var is x, y, or z. In other words, we have terms:

$$z = xy \text{ or } x/y \qquad g_z = \frac{\Delta z}{z} \qquad g_x = \frac{\Delta x}{x} \qquad g_y = \frac{\Delta y}{y}$$

Note that using these definitions, we can write z between two periods in time z_t and z_{t+1} as the following:

$$z_{t+1} = z_t + \Delta z = z_t\left(1 + \frac{\Delta z}{z_t}\right) = z_t\left(1 + g_z\right)$$

where (for now) we assume that the percent change is the same every period. Similar expressions can be written for x and y.

A bit of terminology here that I may use is the following:

- **Growth Rate**: $g_z = \frac{z_{t+1} - z_t}{z_t} = \frac{\Delta z}{z} = \left(\frac{z_{t+1}}{z_t} - 1\right)$
  - Note that g_z here is given in decimal terms. You may be more used to g_z in percent terms, which would just be the g_z here times 100.
- **Growth Factor**: $(1 + g_z)$
  - The "growth factor" is given this name because it is the factor you use to multiply one period to get to the next period. As we can see in the equation above, we have $z_{t+1} = (1 + g_z)\, z_t$

The expression in terms of percent change, combined with the definition of $z = xy$ gives us:

$$\frac{z_{t+1}}{z_t} = \frac{x_{t+1}y_{t+1}}{x_t y_t} = \left(\frac{x_{t+1}}{x_t}\right)\left(\frac{y_{t+1}}{y_t}\right)$$
$$1 + g_z = (1 + g_x)(1 + g_y)$$

Similarly, when we have $z = x/y$ then the formula becomes:

$$\frac{z_{t+1}}{z_t} = \frac{(x_{t+1}/y_{t+1})}{(x_t/y_t)} = \frac{(x_{t+1}/x_t)}{(y_{t+1}/y_t)}$$
$$1 + g_z = \frac{(1 + g_x)}{(1 + g_y)}$$

Then we have the following two useful results that are directly based on the polynomial approximations above. In this case, the g_x and g_y have taken the place of x and y from the earlier equation. This means that the approximations will work well when g_x and g_y are close to zero (in DECIMAL terms).

(7) If $z = xy$ then $g_z \approx g_x + g_y$

(8) If $z = x/y$ then $g_z \approx g_x - g_y$

3 Compound Growth

Suppose we have a variable z_t that takes on a different value every period, for example every year. In addition, between one year and the next there is a growth rate of g_t. If we fix a starting value, for example in the year 2000, $z = z_0$, then we can use the growth rates to calculate the level of z at any point in time as follows:

$$
\begin{aligned}
z_1 &= (1 + g_1)\, z_0 \\
z_2 &= (1 + g_2)\, z_1 = (1 + g_2)\,(1 + g_1)\, z_0 \\
z_3 &= (1 + g_3)\, z_2 = (1 + g_3)\,(1 + g_2)\,(1 + g_1)\, z_0
\end{aligned}
$$

Notice that given the initial value z_0, we can keep using the relationship between z_t and z_{t-1} to get an expression in terms of z_0. If we keep repeating this process, we get a formula for any z_t in terms of z_0 and the intervening one-period growth rates:

$$z_t = (1 + g_t)\, z_{t-1} = \left[\prod_{i=1}^{t} (1 + g_i)\right] z_0$$

That product term is fairly complicated, and it may be difficult to compare directly to our one-period growth rates. Is there a simpler way to summarize this information? One such simplification would be to ask "what is the single growth rate that, if held constant, would have gotten me from z_0 to z_t in the same amount of time?" In other words, we want to find g^* that solves the equation:

$$z_t = (1 + g^*)^t\, z_0$$

We call this g^* the **compound average growth rate (CAGR)** for z between time 0 and time t. In particular, the "growth factor" associated with the CAGR is a **geometric average** of the growth factors corresponding to the one-period growth rates between 0 and t:

$$\left(\frac{z_t}{z_0}\right)^{\frac{1}{t}} = \left[\prod_{i=1}^{t}(1 + g_i)\right]^{\frac{1}{t}} = 1 + g^*$$

Just like with an arithmetic average, we can calculate the average if we know all the individual values (the second term), or if we know the total value and the number of elements in the average (the first term).[1]

4 Exponents and Logarithms

Here, we review the rules for exponentiation and logarithms. We present the rules here for an arbitrary "base" value b. In economics, we usually focus on the "natural logarithm," written as ln, which refers to the logarithm with the base value e (Euler's number).

[1] In an arithmetic average, we would have $AVG = \frac{2+3}{2} = \frac{5}{2} = 2.5$

In general, the usefulness of exponents and logarithms is that they can greatly simplify calculations. In particular, they transform multiplication into addition, division into subtraction, and exponentiation into multiplication.

4.1 Exponent and Logarithm Rules

First, the exponent rules are as follows:

- $b^x b^y = b^{(x+y)}$
- $(b^x)^y = b^{xy}$
- $\frac{b^x}{b^y} = b^{x-y}$

Two useful cases to keep in mind are the exponent term is 0 or 1:

- $b^0 = 1$
- $b^1 = b$

Next, recall that we can define the logarithm as the inverse of exponentiation. In other words:

$$b^x = c \iff log_b(c) = x$$

In turn, the logarithm rules mirror the exponent rules:

- $log_b(xy) = log_b(x) + log_b(y)$
- $log_b(x^y) = y \times log_b(x)$
- $log_b(x/y) = log_b(x) - log_b(y)$

Similar to exponentiation, we have two special cases that apply for any base value b:

- $log_b(1) = 0$
- $log_b(b) = 1$

4.2 Expressing any number using any base

It is possible to rewrite any number as an equation in terms of exponents and logarithm. This follows from the fact that we defined exponents and logarithms to be inverse operations. Thus, if we combine the two operations we get back to our original value. This gives us:

$$c = b^{log_b(c)} = log_b\left(b^c\right)$$

This is particularly helpful when c is already expressed as an exponential (derivative) equation:

$$x^y = b^{log_b(x^y)} = b^{y log_b(x)}$$

4.3 Application - Compound Growth

Given that macroeconomics is concerned (among other topics) with growth over time, having good tools for dealing with growth is essential. In particular, economies seem to be characterized by a regular *rate* of growth, in other words the RATIO between two periods is (fairly) steady over time:

$$\frac{z_t}{z_{t-1}} = 1 + g_t \rightarrow log\left(\frac{z_t}{z_{t-1}}\right) = log(z_t) - log(z_{t-1}) = log(1 + g_t)$$

When we look at changes over longer periods of time, we are looking at the compound growth formula from earlier. Using logs will convert the product term (which may be difficult to calculate) into a sum:

$$log\left(\frac{z_t}{z_0}\right) = log\left(z_t\right) - log\left(z_0\right) = log\left(\prod_{i=1}^{t}(1 + g_i)\right) = \sum_{i=1}^{t} log\left(1 + g_i\right)$$

Similarly, we can also get an expression in terms of the compound average growth rate (CAGR) g^* based on the formula from earlier:

$$\begin{aligned} log\left(\frac{z_t}{z_0}\right) &= log\left(\left[\left(\frac{z_t}{z_0}\right)^{1/t}\right]^t\right) \\ &= t \times log\left(\left[\frac{z_t}{z_0}\right]^{1/t}\right) \\ &= t \times log\left(1 + g^*\right) \end{aligned}$$

Notice that given a constant growth rate g^*, the log-level of z_t can be drawn as a straight line over time. In many cases, linear functions are much easier to work with, highlighting one of the main advantages of working with logs.

The log equation also gives us a convenient way to see how CAGR is related to the growth rate in each period. The log of the CAGR growth factor $(1 + g^*)$ can be expressed as the average of the log-growth factors $(1 + g_i)$ since we have:

$$\begin{aligned} log\left(\frac{z_t}{z_0}\right) &= \sum_{i=1}^{t} log\left(1 + g_i\right) = t \times log\left(1 + g^*\right) \\ \frac{1}{t}\sum_{i=1}^{t} log\left(1 + g_i\right) &= log\left(1 + g^*\right) \end{aligned}$$

Finally, note that we can express the initial compound growth equation by switching the base in the logarithm. Specifically, we have:

$$\begin{aligned} z_t &= (1 + g^*)^t z_0 \\ z_t &= b^{t\ log_b(1+g^*)} z_0 \end{aligned}$$

As we explain in more detail below, the most common choice of base is to use e (Euler's Number). This gives us the following expression:

$$\begin{aligned} z_t &= e^{t\ln(1+g^*)} z_0 \\ &\approx e^{tg^*} z_0 \end{aligned}$$

Thus, the equation $z_t = e^{tr} z_0$ is just a compact (and approximate) way of writing $z_t = (1+r)^t z_0$.

4.4 Natural Logs and Percent Change

So far we've discussed properties of logarithms that apply regardless of the choice of the "base." Is there a reason to prefer one base over another? In economics we usually choose to focus on the natural logarithm, or log-base e. In this class, the main reason is that there is a close numerical relationship between the natural logarithm and percent changes. In particular, when g is close to zero we have the following approximation:

$$log_e(1+g) = \ln(1+g) \approx g$$

This is particularly helpful because it gives a much more intuitive way of discussing equations and numeric results based on equations using logarithms.

An equation of this form arises very naturally when we look at growth over time, as we saw earlier. To make this connection a bit more explicit, let's look at the log-difference between two consecutive periods:

$$\begin{aligned} \ln(z_t) - \ln(z_{t-1}) &= \ln\left(\frac{z_t}{z_{t-1}}\right) \\ &= \ln\left(\frac{z_{t-1} + \Delta z}{z_{t-1}}\right) = \ln\left(1 + \frac{\Delta z}{z_{t-1}}\right) \\ &= \ln\left(1 + g_z\right) \approx g_z \end{aligned}$$

where g_z is the growth rate of z in decimal terms.

Using the natural log approximation also further simplifies the relationship between the compound average growth rate and the one-period growth rates:

$$\frac{1}{t}\sum_{i=1}^{t} \ln\left(1+g_i\right) = \ln\left(1+g^*\right)$$

$$\frac{1}{t}\sum_{i=1}^{t} g_i \approx g^*$$

This tells us that, when the growth rate each period is small, the compound average growth rate is approximately the simple (arithmetic) average of the one-period growth rates.

So far, we've just asserted that this is a good approximation. However, let's take a look at some sample values to see if this approximation works well in the range of values we see in economic data. Recall that we are using decimal units for x; for example, a growth rate of 4% corresponds to $x = 0.04$ and a growth rate of -2% corresponds to $x = -0.02$. In the table below, we can see that the approximation error $(x - \ln(x))$ is less than half a percentage point (0.005) up to about 10% ($x = 0.1$). For negative growth, the error grows modestly faster but is still relatively small.

	Positive Growth (x)		**Negative Growth ($-x$)**	
x	$\ln(1+x)$	$x - \ln(1+x)$	$ln(1-x)$	$(-x) - ln(1-x)$
0.00	0.0000	0.0000	0.0000	0.0000
0.02	0.0198	0.0002	-0.0202	0.0002
0.04	0.0392	0.0008	-0.0408	0.0008
0.06	0.0583	0.0017	-0.0619	0.0019
0.08	0.0770	0.0030	-0.0834	0.0034
0.10	0.0953	0.0047	-0.1054	0.0054
0.12	0.1133	0.0067	-0.1278	0.0078
0.14	0.1310	0.0090	-0.1508	0.0108

Overall, this approximation seems to work well for the growth rates typically seen in macroeconomic data. Very fast-growing economies can sustain growth rates of 10% or more for a decade or longer[2]. However, these are somewhat exceptional experiences — in 2016 the International Monetary Fund (IMF) estimated that only 3 countries (Iran, Iraq, Nauru) had annual growth of 10% or more.[3] Estimates from the Bureau of Economic Analysis (BEA) show that the fastest annual real GDP growth rate in the United States since 1960 was 7.3% (in 1984). Between 1960 and 2017, U.S. growth has averaged 3.0% annually.[4]

It is also notable that the measurement error in GDP is quite large. While we don't have a true value to compare against, two guides for the extent of measurement error are (1) comparing GDP across revisions (i.e. updates to GDP estimates as more and better data becomes available) and (2) comparing GDP to GDI (values which, in theory, should be the same). For both revisions and comparing GDP and GDI it is not uncommon for the values to differ by as much as 1 percentage point (e.g. 2% vs 3%). Thus, the error due to using a

[2] Examples include China in the 1990s and 2000s, Asian NICs (Hong Kong, Singapore, South Korea, and Taiwan) during the 1970s-1990s.

[3] See the IMF World Economic Outlook (WEO) database for October 2017: https://www.imf.org/external/pubs/ft/weo/2017/02/weodata/index.aspx

[4] BEA data can be accessed at https://www.bea.gov/national/index.htm

log approximation is small relative to the inherent uncertainty in our estimates.[5]

5 Functions

A function is a general term for a rule that takes in one or more inputs and assigns an output. In particular, with a function we are guaranteed that when we are given the same input we get the same output.

One of the major questions we might have about a function is how does the value of the function (the output) change when we change the inputs? For example, if we have a function that relates aggregate consumption to GDP:

$$C = f(Y)$$

we can ask "does increasing GDP lead to higher consumption?" For a given function, we can ask this question in three ways:

- Does increasing GDP lead to higher consumption?
- Is $C = f(Y)$ an increasing function of Y?
- Is the derivative of $f(Y)$ positive? Also written as $f'(Y) > 0$ or $\frac{d}{dx}f(x) > 0$.

When we have multiple inputs to the function, we can rephrase the third option in terms of the partial derivative. Recall that when we take a partial derivative we treat the other variables as fixed. The notation for partial derivative is:

- Partial derivative of $f(Y, X)$ with respect to Y: $\frac{\partial}{\partial Y}f(Y, X)$ or $f_1(Y, X)$

While the partial derivative is a convenient tool for analyzing our models, it is not obvious that we actually observe a partial derivative in the real world. The imaginary exercise of only change Y almost never happens in real life; all sorts of stuff differs across individuals, across time, and across countries. Ever worse, this unnamed X variable may be something that changes over time and that we aren't able to measure very well (for example, consumer expectations).

5.1 Linear Function - Basics

Usually, we describe a linear function as follows:

$$y = a + bx$$

[5] It should be noted that this isn't a perfect comparison, since measurement error is in absolute terms (i.e. it can be positive or negative in a given period) whereas the approximation error always leans in one direction.

where a is called the **y-intercept** and b is called the **slope** of the line. Graphically, a is where the line crosses the y axis and b controls how steep the line is.

If we think that this relationship holds at every point in time, we can use this "level" relationship to construct another equation "in changes" as follows:

$$\begin{aligned} y_1 &= a + bx_1 \\ y_0 &= a + bx_0 \\ (y_1 - y_0) &= b\,(x_1 - x_0) \\ \Delta y &= b\,(\Delta x) \end{aligned}$$

In this case, we can see that the slope of the line is another way of saying what a change in x implies for a change in y. Specifically, if x changes by 1 unit then y changes by b units. Notice also that the equation in terms of changes is ITSELF a linear equation, just with new names for the variables. Specifically, $y \to \Delta y$, $x \to \Delta x$, $a \to 0$ and $b \to b$.

The distinctive characteristic of a "linear" equation is that this relationship between changes in the x and y variables is constant (since b is fixed). This corresponds to the fact that the derivative of this expression $\frac{dy}{dx} = b$ is a constant. In addition, a linear function means that changes larger changes in x scale directly into changes to y. In other words, if Δx is twice as big, then Δy is also twice as big.

In general, we can also write a linear function with many right-hand side values:

$$y = a + b_1x_1 + b_2x_2 + b_3x_3$$

The key is that the a and b values are fixed even when the variables x and y change.

5.2 Linear Equations with Logs

When interpreting any equation, it is important to keep in mind what the x and y values stand for. For example, earlier we showed an equation in terms of logarithms that had the $y = a + bx$ form. Specifically, we had:

$$\ln(z_t) = \ln(z_0) + t \times \ln(1 + g^*)$$

Let's compare the variables in this equation to the $y = a + bx$ form. We have $a = \ln(z_0)$ and $b = \ln(1 + g^*)$ together with $y = \ln(z_t)$ and $x = t$. Given that we are working with logarithms, we can interpret the slope $\ln(1 + g^*)$ as the (average) growth rate of z_t for each unit of time. Specifically, each period ($\Delta t = 1$), the y variable $\ln(z_t)$ will increase by $\ln(1 + g^*)$. As we showed earlier, this also means that z_t will increase by approximately $100 \times g^*$ percent.

As we noted earlier, we can also use logarithms to turn multiplication into addition and exponentiation into multiplication. The major benefit of this is that we can take something that is not a linear function and turn it into something that IS linear. In addition to their general ease of working with, having equations in a linear form means we can use the regression techniques below to estimate parameters.

A few examples of how to turn use logs to generate a linear relationship follow:

$$z = x^{\alpha}y^{1-\alpha} \rightarrow \ln(z) = \alpha\ln(x) + (1-\alpha)\ln(y)$$
$$z = e^{5x+3} \rightarrow \ln(z) = 5x + 3$$

Recall that whenever we talk about a linear equation in terms of x and y, we are doing so in a general way. Depending on the context, you may need to think carefully about what the "x" and "y" from the general form of a linear equation mean for the equation you are looking at.

5.3 Warning on Interpretation

In this class, we will try to push back against a conventional interpretation of these linear relationships. The order of the variables in this equation (and the choice of horizontal and vertical axes on a graph) is usually meant to suggest that y is the "dependent" variable and x is the "independent" variable. Other common terminology is that x "explains" y, or even that x CAUSES y. While this may in fact be true, just plotting some data and drawing a line does not make it so.

6 Appendix A - Taylor Series and Approximation for $\ln(1 + x)$

Taylor's Theorem, a famous result from calculus, tells us that we can use the derivative of a function at a point a to approximate the value of that function for small movements away from a. As the movement away from a gets smaller, the approximation gets better. In the limit, as we get closer and closer to a, the value of the function and the value of the approximation coincide again. You will cover this theorem if you take a class in calculus or in real analysis.

Rather than prove Taylor's Theorem from scratch, it is enough for our purposes to use the first order Taylor expansion, that is the approximation around a point a using only the first derivative at that point. The approximation is given as:

$$f(x) \approx f(a) + f'(a)(x - a)$$

where x is the point after deviation from a, and a is the point we are trying to approximate around.

In our case we want to approximate $\ln(x)$ for movements away from $\ln(1)$. That is, $a = 1$, $x = 1+x$ and $f = \ln$. Evaluating the log away from 1 can be tricky, but at 1 the log couldn't be simpler. First, we know that $\ln(1) = 0$. Second, we know that the derivative of ln is $1/x$, so at $x = 1$, we have $f'(1) = 1/1 = 1$. Plugging all this back in, we have:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ \ln(1+x) &\approx \ln(1) + \ln'(1)(1+x-1) \\ \ln(1+x) &\approx 0 + 1(x) \\ \ln(1+x) &\approx x \end{aligned}$$

Using this result on our equation about percent changes (in decimal terms), we have:

$$\ln\left(1 + \frac{\Delta z}{z}\right) \approx \frac{\Delta z}{z}$$

Given that this is an approximation, it is also helpful to think about how this approximation differs from the precise answer. In particular, we know that because log is a concave function (it has a negative second derivative, and it bends towards the x-axis in a graph) then the Taylor approximation will be larger than the actual function. In notation, we would say that:

$$\ln\left(1 + \frac{\Delta z}{z}\right) < \frac{\Delta z}{z}$$

This inequality holds as long as $\Delta z \neq 0$. In particular, this holds for positive and negative values of Δz. As we saw in the table earlier, this means that $\ln(1-x)$ is MORE negative than $-x$.

7 Appendix B - Proof of Appendix 2 Approximations

7.1 Proof using Expansion

Polynomial Approximations: Note that all of these approximations work well only when x and y are close to zero.

(3) $\boldsymbol{(1+x)(1+y) \approx 1 + x + y}$
First, let's multiply out the left-hand side expression and see how it compares to $1 + x + y$.

$$(1+x)(1+y) = 1 + x + y + (xy)$$

Switching to the approximation is arguing that the (xy) term is so small that we can safely ignore it. Let's use a numeric example to see how this works, using economically reasonable numbers like growth rates of 2% and 1.5%. This would give us:

$$\begin{aligned} (1+.02)(1+0.015) &= \left(1 + \frac{20}{1000}\right)\left(1 + \frac{15}{1000}\right) \\ &= 1 + \frac{20}{1000} + \frac{15}{1000} + \frac{(20)(15)}{(1000)^2} \\ &= 1 + .02 + .015 + .0003 \\ &= 1.0353 \end{aligned}$$

Since x and y were close to zero, the denominator in the (xy) term is much larger than the numerator. In this example, if we rounded to the level of precision we had in the initial estimates of growth (to the 1000s digit or to 0.001 or 0.1%) then there is no apparent difference between the exact and the approximate formula.

(4) $\boldsymbol{(1+x)^2 \approx 1+2x}$
This is a straightforward application of result from part (3), just with $y = x$.

(5) $\boldsymbol{(1+x)^n \approx 1+nx}$
This follows from iterating forward from the $(1+x)^2$ case. To be a bit more precise, we would say:

$$(1+x)^n = (1+x)(1+x)^{n-1} \approx (1+x)(1+(n-1)x) = 1+x+(n-1)x+(n-1)x^2 \approx 1+nx$$

Clearly, every time we use this approximation we are dropping additional terms. In this sense, for a given x when we increase n the approximation becomes progressively worse.

(6) $\boldsymbol{\frac{1+x}{1+y} \approx 1+x-y}$
This is somewhat trickier. The approach in the textbook is indirect - suppose that this approximation is valid and then see how large the error is. By multiplying through by $1+y$, we have:

$$1+x \approx (1+x-y)(1+y) = 1+x-y+y+xy-y^2 = 1+x+xy-y^2$$

As we've argued earlier, when x and y are both close to zero, then the product terms (xy and $-y^2$) will be much smaller than either x or y themselves. We can also see what an exact answer would look like:

$$\begin{aligned} \frac{1+x}{1+y} &= 1+c \\ 1+x &= (1+c)(1+y) \\ x &= c+y+cy \\ \frac{x-y}{1+y} &= c \end{aligned}$$

7.2 Proof using Log-Approximation

In the steps below, we apply the Taylor series approximation for natural log, and then adjust the estimate to get back to $(1+x)$. Specifically, we have:

$$\begin{aligned} \ln(1+x) &\approx x \\ 1+\ln(1+x) &\approx 1+x \end{aligned}$$

Recall again that the basis for the approximation is that whatever is inside the log is close to 1. For example, invoking this approximation is claiming that $(1+x)(1+y)-1$ or $(1+x)^n-1$ is close to zero.

(3) $\boldsymbol{(1+x)(1+y) \approx 1+x+y}$

$$(1+x)(1+y) \approx 1+\ln\left[(1+x)(1+y)\right] = 1+\ln(1+x)+\ln(1+y) \approx 1+x+y$$

(4) $\boldsymbol{(1+x)^2 \approx 1+2x}$

$$(1+x)^2 \approx 1+\ln\left[(1+x)^2\right] = 1+2\ln(1+x) \approx 1+2x$$

(5) $\boldsymbol{(1+x)^n \approx 1+nx}$

$$(1+x)^n \approx 1+\ln\left[(1+x)^n\right] = 1+n\ln(1+x) \approx 1+nx$$

(6) $\boldsymbol{\frac{1+x}{1+y} \approx 1+x-y}$

$$\frac{1+x}{1+y} \approx 1+\ln\left[\frac{1+x}{1+y}\right] = 1+\ln(1+x)-\ln(1+y) \approx 1+x-y$$

7.3 Proof using Taylor Approximation

We can use the Taylor series methods we discussed earlier in the case of natural log and apply them to the polynomials examined in Appendix 2. The main wrinkle is that for some of these exercises we need to use a multivariable version of the Taylor formula because we have to look at both x and y changes. This formula is a fairly intuitive extension of the single-variable Taylor formula, and is as follows:

$$f\left(x,y\right) \approx f\left(x_0,y_0\right) + f_1\left(x_0,y_0\right)\left(x-x_0\right) + f_2\left(x_0,y_0\right)\left(y-y_0\right)$$

where (x_0, y_0) refers to some initial point (analagous to a from earlier) and where $f_1(x_0, y_0)$ and $f_2(x_0, y_0)$ refer to the partial derivative of f with respect to x and y (respectively), evaluated at the point (x_0, y_0).

In the proofs below, we will focus on the point $x_0 = 0$ or $(x_0, y_0) = (0, 0)$.

(3) $\boldsymbol{(1+x)(1+y) \approx 1+x+y}$
In this case, we have $f(0,0) = (1+0)(1+0) = 1$. In addition, the partial derivatives are $f_1(x_0, y_0) = (1+y_0) = 1$ and $f_2(x_0, y_0) = (1+x_0) = 1$. Finally, putting this into the Taylor formula for the multivariable case, we have:

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0) \\ (1+x)(1+y) &\approx 1 + (1)(x-0) + (1)(y-0) = 1+x+y \end{aligned}$$

(4) $\boldsymbol{(1+x)^2 \approx 1+2x}$ (see next proposition)

(5) $\boldsymbol{(1+x)^n \approx 1+nx}$
In this case, we only need to work with a single variable. This gives us $f(0) = (1+0)^n = 1$ and the derivative as $f'(0) = n(1+0)^{n-1} = n$. Plugging this into the single-variable Taylor formula, we have:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ (1+x)^n &\approx 1+nx \end{aligned}$$

(6) $\boldsymbol{\frac{1+x}{1+y} \approx 1 + x - y}$

Returning again to the 2-variable case, we have $f(0,0) = (1+0)(1+0) = 1$. The partial derivatives are a bit different this time, however, giving us $f_1(x,y) = \frac{1}{1+y}$ and $f_2(x,y) = \frac{-(1+x)}{(1+y)^2}$. Plugging this into the formula, we have:

$$f(x,y) \approx f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0)$$

$$\frac{(1+x)}{(1+y)} \approx 1 + (1)(x-0) + (-1)(y-0) = 1 + x - y$$

8 Appendix C - Proof of Geometric Series Formula

Let's call the sum of the first n terms of the geometric series $S(n)$. In other words:

$$S(n) = \sum_{i=0}^{n} r^i$$

Next, we can rearrange the terms to show how $S(n)$ and $S(n+1)$ relate to each other in a straightforward way. Obviously, we know that:

$$S(n+1) - S(n) = \left[\sum_{i=0}^{n+1} r^i\right] - \left[\sum_{i=0}^{n} r^i\right] = r^{n+1}$$

However, there is a second rearrangement of $S(n+1)$ that will make it a function of $S(n)$:

$$\begin{aligned} S(n+1) &= \sum_{i=0}^{n+1} r^i = 1 + \sum_{i=1}^{n+1} r^i \\ &= 1 + r\sum_{i=1}^{n+1} r^{i-1} = 1 + r\sum_{k=0}^{n} r^k \\ &= 1 + rS(n) \end{aligned}$$

Note that the second equality on the 2nd line is just a change of variables where $i - 1 = k$, and adjusting the starting and stopping points of the summation from $i = 1 \to k = 0$ and $i = n+1 \to k = n$. Combining these two equations, we have:

$$\begin{aligned} S(n+1) - S(n) = 1 + rS(n) - S(n) &= r^{n+1} \\ (r-1)S(n) &= r^{n+1} - 1 \\ (1-r)S(n) &= 1 - r^{n+1} \\ S(n) &= \frac{1-r^{n+1}}{1-r} \end{aligned}$$

For the infinite limit, we can see that if $|r| < 1$ that as n gets arbitrarily large the r^{n+1} term in the numerator goes to 0 and we are left with $\frac{1}{1-r}$. If $|r| > 1$ then the expression explodes since r^{n+1} will become arbitrarily large. If r is negative, it also alternates between (very large) positive and negative values. When $r = 1$, the expression shown here is not defined (since (1-1) = 0 which would be an invalid division). However, we know that the sum of 1s grows arbitrarily large. When $r = -1$ the sum alternates between 0 (n is even) and 1 (n is odd).