Leverage and Disagreement*

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Abstract

I build a model of the cross section of leverage ratios for borrowers based on heterogenous beliefs about future asset returns and endogenous collateral constraints. In equilibrium, borrowers and lenders are matched in an assortative way according to their relative optimism through hedonic interest rates, which are disconnected from risk aversion or expected default probabilities. Under minimal assumptions on the underlying distribution of beliefs, the leverage ratio distribution of borrowers is a truncated Pareto with coefficient 2 in the upper tail, a prediction verified in micro-level administrative data for homeowners, publicly available data for entrepreneurs and in the TASS database for hedge funds. Expected and realized returns to levered portfolios are very skewed and fat tailed, even when heterogeneity in beliefs vanishes. The market features a high degree of customization and fragmentation, as many real world financial markets which are organized over-the-counter. Pyramiding lending arrangements (equivalently tranching) result from the desire of lenders to lever into these allocative interest rates; extended with short-sales, the model provides an equilibrium characterization of short interest and rebate rates. Finally, the leverage ratio distribution of borrowers provides useful information on the buildup of risk, an insight I illustrate using data for the US housing market between 1987 and 2012.

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Introduction

Belief disagreement has been blamed for the importance and severity of the last financial crisis. Optimistic about the future evolution of house prices, many homeowners bought houses using leverage, and some banks levered into Mortgage-Backed Securities that would only pay off if house prices continued to rise. When house prices decreased, Lehman Brothers filed for bankruptcy, and other major banks and insurance companies would have gone bankrupt without government intervention. Why did banks take on such a levered bet on future house price values? Can regulators devise early-warning systems alerting in real time on the buildup of risk? This paper contributes new answers to these questions.

In this paper, I develop a model of levered investing driven by heterogeneous beliefs about the future payoff of a risky asset and endogenous collateral constraints. Under minimal assumptions on the underlying distribution of beliefs, the leverage ratio distribution of borrowers is a truncated Pareto in the upper tail. The Pareto tail coefficient on final borrowers' leverage ratio distribution is informative of the number of levels of pyramiding lending arrangements driven by heterogenous beliefs, and therefore of the relative optimism of the marginal investor. The leverage ratio distribution thus could be used to inform regulators on the degree of asset price overvaluation: in particular, the distribution of credit is much more informative than the aggregate level of credit itself. Developing a model of the leverage ratio distribution turns out to be also useful to get a new perspective on interest rates, pyramiding lending arrangements, and short-sales, all of which are central to the understanding of the last financial crisis and levered speculative investing more generally. For example, borrowers and lenders and matched in an assortative way according to their relative optimism through hedonic interest rates, which are disconnected from risk aversion or expected default probabilities. The market features a high degree of customization and fragmentation, as many real world financial markets which are organized Over-The-Counter (OTC). Pyramiding lending arrangements, equivalently tranching, result from the desire of lenders to lever into these allocative interest rates. Extended with short-sales, the model provides an equilibrium characterization of short interest and rebate rates.

The model is set in two periods with a continuum of risk-neutral agents who have heterogenous beliefs about the payoff of a risky asset. The simplest version of the model rules out short-sales and pyramiding lending arrangements, but both restrictions are relaxed in extensions. In the baseline model, agents can invest their endowment in three ways. They can engage in levered investing, offer collateralized lending to investors in the risky asset, or invest in an alternative (non disagreement) technology. In equilibrium, the population of agents partitions into three groups endogenously based on their level of optimism: the most optimistic do levered investing, agents with intermediate levels of optimism do collateralized lending, and the least optimistic invest in the alternative technology. Agents are generally reluctant to lend, as they worry about low collateral values upon default. As a result, they set endogenous borrowing limits that reflect their relative pessimism about collateral values, as in Geanakoplos (1997) or Simsek (2013).

A key new ingredient in the present model is that there is more than one lender, and that in general different lenders have different levels of optimism towards the asset. More optimistic lenders allow borrowers to achieve a higher leverage ratio, because they agree to lend more for each unit of risky asset in collateral. But buying more units of the risky asset is worth more to relatively more optimistic borrowers, creating a complementarity between borrowers' and lenders' beliefs. In equilibrium, competition leads to positive sorting between borrowers and lenders, as more optimistic lenders lend to more optimistic borrowers. Higher leverage loans would be attractive to all borrowers if implicit returns on loan contracts were the same, because all risk-neutral borrowers would ideally like to lever themselves into the asset as much possible. For markets to clear, interest rates must therefore rise when leverage increases, so that only more optimistic borrowers are willing to lever more. Interest rates are hedonic prices, and are higher than the returns to cash, even though loans are perfectly safe according to lenders and borrowers trading them. Moreover, each lender is effectively lending to one particular borrower using a contract with a different leverage ratio and a different interest rate, and the sorting occurs through the loan contracts that both lenders and borrowers endogenously choose. It looks as if trade was bilateral and exhibited price dispersion, even though markets are walrasian and the terms of trade are set competitively. This may be why many of these markets are OTC.

Because lenders have different degrees of optimism, loans with many different leverage ratios coexist in equilibrium for the same asset. For small disagreement, and under very mild assumptions on the structure of beliefs, the distribution of borrowers' leverage ratios is a truncated Pareto distribution. This Pareto distribution for leverage ratios obtains in this model through a microfounded and static mechanism, unlike in existing models of random growth. The reason why some borrowers can borrow with very high leverage ratio loans is that the corresponding lenders are close to being marginal investors in the asset, for whom the price is almost right, and who grant a loan amount almost equal to the price of the asset itself. When disagreement becomes very small, the leverage of the most optimistic borrowers goes to infinity because lenders are less and less worried about the collateral. This effect is stronger than the diminished room for speculation allowed by lower disagreement. To the limit, very few borrowers end up borrowing almost all agents' wealth, and the distribution of their leverage ratios becomes a full (untruncated) Pareto.

Moreover, expected and realized returns to levered investing are showed to be potentially very large, fat tailed and skewed, regardless of the underlying level of disagreement. They both depend on the true realization of the asset in the second period, and the amount of competition for leverage and the asset in the first period. When the return realization is inside the set of beliefs of borrowers, there is a non monotonic relationship between leverage and returns. The highest return is always achieved by the agent who was exactly right about the value of the asset in period one.

I consider two extensions of this model. First, I allow for collateralized loans to themselves be used as collateral, for example in a repo transaction, or through rehypothecation. When available, this possibility is used in equilibrium by direct lenders to borrowers: lenders want to lever up into hedonic interest rates, which are higher than the returns to cash. These "pyramiding lending arrangements" are shown to be equivalent to the tranching of assets, where collateral is used multiple times, to back multiple promises. The tail coefficient on borrowers' leverage ratio distribution decreases, as final borrowers' leverage results both from their own borrowing from lenders, but also from their lenders' borrowing. At the same time, the price of the risky asset is then shown to reflect even more the opinion of the most extreme optimists, as more funds are channeled towards the risky asset thanks to these pyramiding arrangements. Moreover, expected returns of final borrowers increase considerably.

In a second extension, I allow for the short-selling of assets while maintaining the possibility for all agents to borrow and lend freely, unlike in Simsek (2013). Asset prices this time reflect the opinion of the most pessimistic agents. This extension provides another new intuition for why rates on fixed income securities may not be determined solely by expected default probabilities: lenders are natural short-sellers of the asset, and therefore borrowers need to give them a return in equilibrium in so that they prefer to lend rather than short. Endogenous rebate rates arise, not as physical costs of short-selling, but again as competitive assignment prices which match the most pessimistic short-sellers to the most pessimistic asset lenders. Only a small fraction of assets is on loan in equilibrium, even though all are available ex-ante. Short interest is endogenous in the model, in the order of a few percentage points, as in reality.

Finally, some potential applications of the model are illustrated in the last part of the paper. Using Dataquick data from 1987 to 2012, I show through the lens of the model how the evolution of the leverage ratio distribution of US homeowners was very strongly suggestive of the buildup of risk in the financial system between 2000 and 2006, with more and more pyramiding lending arrangements. With the benefit of hindsight, this likely was the signature of a growing shadow banking sector, the extent of which took economists by surprise in 2008. Monitoring final borrowers' leverage ratio distribution may have helped avoid that surprise.

The distribution of hedge funds' leverage ratios, as well as that of entrepreneurs, are two other potential applications of this model of levered investing, and are both well approximated by Pareto distributions. The hedge funds application is illustrative of the fact that when disagreement is very low (for example on arbitrage strategies), competition leads naturally to very large and risky positions. It is also representative of how such highly levered strategies can lead to very fat tailed and skewed ex-post returns, which are well approximated by Pareto distributions, however small the underlying level of belief heterogeneity. Proprietary trading in investment banks and hedge funds generate very heterogenous incomes, and so does entrepreneurship. The present paper presents a new mechanism why this may be so.

The rest of the paper proceeds as follows. Section 1 presents the simplest version of the model, where borrowing contracts cannot be used as collateral, and short-selling is impossible. Section 2 presents the properties of this model. Section 3 extends the baseline model to the case of pyramiding lending arrangements or tranching. Section 4 allows for unrestricted short-selling of the asset. Section 5 shows three potential empirical applications of the framework, with three different types of borrowers: homeowners, entrepreneurs, and hedge funds. Section 6 concludes.

Literature

This paper is part of a large literature investigating the consequences of belief disagreement in financial markets. Miller (1977) was perhaps the first to highlight that because of short-sales constraints, asset prices can be higher than mean beliefs because pessimists cannot express their negative opinions. Dynamic versions of this model, such as Scheinkman and Xiong (2003), embed a resale value option, following Harrison and Kreps (1978). These models have been used to explain the dotcom bubble episode (Ofek and Richardson (2003)). However D'Avolio (2002) documents that short-sales costs are actually not so large, of the order of magnitude obtained theoretically in this paper.

This paper follows the pioneering work of Geanakoplos (1997), Geanakoplos and Zame (2002), and Geanakoplos (2003) in embedding these insights in a model with endogenous collateral constraints, where optimists must borrow from lenders who have different beliefs and are therefore reluctant to lend. In Geanakoplos (1997) as well as subsequent models, there is however only one leverage ratio for collateralized loans, because all agents agree on the value of the asset conditional on default. In Simsek (2013), there is also one leverage ratio only, because there are two agents, and therefore one lender; his contribution is to show that the optimism about the probability of upside states has more effect on asset prices than that on downside states.

More broadly, this paper is part of a larger literature investigating the effect of borrowing constraints on asset prices and investment. In this paper, the source of different valuations is belief disagreement, while it is due to different second best uses in some theories of fire sales, like Shleifer and Vishny (1992), Shleifer and Vishny (1997) or Kiyotaki and Moore (1997). In particular, the reason why borrowers need capital is that lenders have more pessimistic beliefs, and hence want to be protected if there is default. There are of course many other theories of borrowing limits, that stem from information asymmetries, lack of commitment, or exogenously imposed margin requirements: Holmstrom and Tirole (1997), Holmström and Tirole (1998), Bernanke and Gertler (1999), Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), among many others. To the best of my knowledge, none of these theories however has predictions about the Power law distribution of leverage ratios. They are also silent on the reasons for pyramiding lending arrangements. I show that these pyramiding lending arrangements arise naturally with heterogeneous beliefs.

Finally, the paper shows a new way to generate Pareto distributions. In this paper, Pareto distributions are generated through a microfounded mechanism linked to an assignment models with complementarities which amplifies even tiny ex-ante differences, unlike in dynamic models of random growth (a survey of which is given in Gabaix (2009)).

1 Borrowing Economy

This section presents the main results of the paper in the simplest possible economic environment. I consider an Economy $\mathcal{E}_{\mathcal{B}}$ where the only available instruments are equity and debt contracts. In particular, I assume that agents cannot sell the real or financial assets short; and that loans cannot be used as collateral. I allow for pyramiding lending arrangements in Section 3, and for short-sales in Section 4. I first present the set-up, then discuss the assumptions, and finally solve for the equilibrium.

1.1 Setup

There are two periods 0 and 1. To simplify notations, I omit the time zero subscript: for example, the price at time 0 is denoted by p instead of p_0. There is a continuum of agents $i \in [0,1]$ of measure one born in period 0 with initial wealth normalized to 1. Agents care only about their consumption in period 1, and therefore need assets to store their wealth.

Assets. To transfer wealth into period 1, agents can invest in a *Storage Technology* with return normalized to $R = 1$. For concreteness, I refer to this non-disagreement asset as *Cash*. They can also invest in an asset in finite supply normalized to 1, with exogenous resale value p_1 in period 1, and endogenous price p in period 0, which I will refer to as the *Real Asset* in the following. In addition, they can agree to collateralized Borrowing Contracts with each other. Formally, I define a Borrowing Contract in Economy $\mathcal{E}_{\mathcal{B}}$ as follows.

Definition 1. (Borrowing Contract, Economy $\mathcal{E}_{\mathcal{B}}$) A *Borrowing Contract* (ϕ) in the Borrowing Economy $\mathcal{E}_{\mathcal{B}}$ is a promise of $\phi \geq 0$ units of Cash in period 1, the *face value*, collateralized by one unit of Real Asset.

Without loss of generality, the set of contracts is restricted to a set containing contracts using exactly one unit of Real Asset as collateral, and that this allows to name contracts by their face value (ϕ). Contracts are traded in an anonymous market at competitive price $q(\phi)$, and payment is only enforced by the collateral: agents default as long as the value of the collateral is lower than the face value of the loan they have to repay. The payoff of contract (ϕ) in period 1 is therefore:

$$\min\{\phi, p_1\}$$

for a contract with face value ϕ. In period 0, this contract is sold by the borrower, who gets $q(\phi)$ units of Cash in exchange for the contract. The interest rate on this borrowing contract is:

$$r(\phi) = \frac{\phi}{q(\phi)}.$$

Beliefs. Agents have heterogeneous point expectations about the price of the asset in period 1. Namely, agent $i \in [0, 1]$ believes that the asset price will be p_1^i with probability 1. This assumption of point expectations may seem extreme. However the model generalizes very straightforwardly to a case where agents are risk neutral and agree about a probability distribution for p_1 around this mean. In particular, in that case, all Borrowing Contracts will in equilibrium be indexed by these states, in which the payoff of the asset differs from its expected mean by an amount that everybody agrees on. The key is therefore that I focus on disagreement about means of future asset payoffs rather than about probabilities of certain events, as in the previous heterogenous beliefs and endogenous margins literature. (which also has risk neutrality) Finally, note that I do not need risk neutrality in the case where agents have point expectations, as for those agents, speculation entails no risk.

More precisely, the cumulative distribution function representing the number of agents with beliefs p_1^i for future prices is denoted by $F(.)$, with corresponding density $f(.)$. The upper bound on agents' beliefs is assumed to be 1 without loss of generality.[1]

[1]The model being linear, all quantities in the model are multiplied by M in the case where this maximum belief is M.

The most pessimistic agents have beliefs $1 - \Delta$, with $\Delta > 0$. $f(.)$ has full support on $[1 - \Delta, 1]$, so Δ is a natural measure of belief heterogeneity. Δ will be referred to as the *belief heterogeneity parameter*.

Note that the assumption made earlier of equal endowment for all agents is also without loss of generality. If initial endowments were heterogenous, then the density function would subsume both how many agents have those beliefs and how wealthy they are.

Equilibrium. All units of the Real Asset are initially endowed to unmodeled agents who sell their asset holdings in period 0 and then consume: for example, the model has overlapping generations and the old sell their holdings to the young before they die. Agent i chooses his position in the Real Asset n^i_A, a menu of financial Borrowing Contracts (ϕ) denoted by $dN^i_B(\phi)$ (where $N^i_B(\phi)$ is the cumulative measure of contracts with face value less then ϕ), and Cash n^i_C, in order to maximize his expected wealth (W) in period 1 according to his subjective beliefs p^i_1 about the Real Asset, subject to his budget constraint (BC), and the collateral constraint (CC):[2]

$$\max_{(n^i_A, dN^i_B(.), n^i_C)} \; n^i_A p^i_1 + \int_\phi \min\{\phi, p^i_1\} dN^i_B(\phi) + n^i_C \tag{W}$$

$$\text{s.t.} \quad n^i_A p + \int_\phi q(\phi) dN^i_B(\phi) + n^i_C \leq 1 \tag{BC}$$

$$\text{s.t.} \quad \int_\phi \max\{0, -dN^i_B(\phi)\} \leq n^i_A \tag{CC}$$

$$\text{s.t.} \quad n^i_A \geq 0, \quad n^i_C \geq 0$$

Note that this portfolio problem is subject to the additional restriction that $n^i_A \geq 0, n^i_C \geq 0$: agents have to choose positive amount of Real Asset and Cash holdings (again, this is relaxed later in the paper). When $dN^i_B(\phi) > 0$, agent i buys Borrowing Contract (ϕ), and therefore lends. When $dN^i_B(\phi) < 0$, agent i sells Borrowing Contract (ϕ), and therefore borrows. Each time a borrower sells of unit of Borrowing Contract, he needs to own unit of asset hence equation (CC) must hold. The equilibrium concept is that of a *Collateral Equilibrium*, as defined in Geanakoplos (1997), with contracts being treated as commodities. Formally,

Definition 2. [Competitive Equilibrium of $\mathcal{E}_\mathcal{B}$] A *Competitive Equilibrium* for Economy $\mathcal{E}_\mathcal{B}$ is a price p for the Real Asset and a distribution of prices $q(.)$ for all traded Borrowing Contracts (ϕ), and portfolios $(n^i_A, dN^i_B(.), n^i_C)$ for all agents i in the Real Asset, Borrowing Contracts and Cash, such that all agents i maximize expected wealth

[2] Since agents trade only one contract in equilibrium, the distribution represented by $N^i_B(\phi)$ has a Dirac density.

in period 1 (W) according to their subjective beliefs, subject to their budget constraint (BC), their collateral constraint (CC), and markets for the Real Asset and Borrowing Contracts clear

$$\int_i n_A^i di = 1, \tag{MC_A}$$

$$\text{and} \quad \forall \phi, \quad \int_i dN_B^i(\phi) di = 0. \tag{MC_B}$$

1.2 Discussion

Before studying the equilibrium of this model, it is worth discussing some of the assumptions I have so far made. This also leads to go through the potential applications of the theoretical model presented above.

Agreeing to disagree. A key assumption in the model is that agents agree to disagree. This disagreement does not come from information, as information alone cannot generate trade (no-trade theorems), and so agents do not learn from the fact that other disagree with them. The assumption of belief disagreement may seem problematic to some readers. It is certainly very likely that agents would have different priors about the returns of a yet unobserved technological advance (such as the internet in the 1990s). As argued in Morris (1995) and Morris (1996), it is very hard to conceive any criteria - rational or otherwise - that would require traders to have the same prior over the dividends of a yet unobserved asset. This perhaps also would apply to the impact of new subprime arrangements in the years 2001-2006. Moreover, Acemoglu et al. (2016), Borovička (2016) and Cao (2017) provide theoretical mechanisms through which heterogenous beliefs can survive market selection. Furthermore, it is well known that disagreement models are isomorphic to models with noisy information aggregation - that is, to models where agents have heterogenous information, but do not learn perfectly from each other, because noise traders are also trading for random reasons.

The most salient argument in support of assuming agents agree to disagree is certainly that the main insights of the model rely on only an epsilon of disagreement between agents, in particular the Pareto results as well as the positive sorting results. As long as there is even an epsilon of noise on financial markets, then agents will never agree perfectly, even in the long run.

Disagreement on means. This model departs from previous papers in the literature on endogenous leverage, in that I assume disagreement on means of future asset returns, together with more than two agents. The assumption that agents disagree on means actually arises naturally in a number of different existing models: for example, models where agents overestimate the precision of their signal, as Scheinkman and

Xiong (2003), lead to disagreement on mean values of asset prices. These models do not have endogenous leverage, though.

In contrast, in Geanakoplos (1997), there is only one leverage ratio: all agents agree on the value of collateral conditional upon default, and disagree either on the value of collateral conditional on other states occurring or on the probability of different states. In the case of a bond, this means all agents agree on the recovery rate. Such is also the case in Simsek (2013), where there is only two types of traders, optimistic and pessimistic. Therefore there must also be only one type of lender, who uses one type of contract. Interest rates compensate for expected default probabilities. This assumption of disagreement on means combined with that of a continuum of traders are the ones leading to a distribution of equilibrium leverage ratios, as well as to hedonic interest rates. All I need in fact is more than two belief types, and the continuum is an especially tractable way to do it.

Another advantage with working with more than two belief types, is that the model can allow for pyramiding lending arrangements, as well as short-selling at the same time as lending. The treatment of pyramiding lending arrangements with endogenous margins is new. Simsek (2013) has a treatment of short-sales, but the model cannot explain short-sales together with lending and borrowing in equilibrium, as it features only two types of agents. In Simsek (2013), one assumption is that some agents (those who short in equilibrium) are inhibited from lending, which is obviously counterfactual. In contrast, the tractability of the present model allows to have all these types of investment strategies coexisting in equilibrium, while everyone has a possibility to short and lend ex-ante. For the same reason, Simsek (2013) cannot model pyramiding, which was central during the last financial crisis.

No short-sales, no pyramiding. The assumption that short-sales is not possible is arguably a good approximation for the housing market[3], as well as for the financing of newly created firms, whose value is not quoted on financial markets: there is then no publicly observable price which would allow traders to bet against the performance of this firm. But again, most of the insights obtained in Economy $\mathcal{E}_{\mathcal{B}}$ are maintained, if not strengthened, when short-sales are possible, as shown in Section 4. An exception concerns the price of the real asset, which is naturally lower than it is in Economy $\mathcal{E}_{\mathcal{B}}$ since pessimistic agents can now express their views.

I also assume that "pyramiding lending arrangements", or the use of loans as collateral are impossible. Equivalently, the rehypothecation of collateral, or the tranching

[3] An exception of course, was the US housing market in 2006, during which derivative securities on mortgage-backed securities made possible to short housing. However, this crucially relied on loans being made to subprime borrowers, the loan being non recourse, and other conditions which are only rarely present simultaneously. Moreover, note that housing was not shorted directly, but loans to homeowners were. **?** is a narrative account of how some investors were able to short the housing market in 2007.

of loans, which are all equivalent in this model, are not not feasible either. Why all these are equivalent and an investigation of what happens when they are feasible is the subject of Section 3.

The real asset. There are many different possible interpretations as to what the real asset, borrowers and lenders refer to. A homeowner could be investing in a house, borrowing from a bank. An entrepreneur could be investing in a project, also borrowing from a bank. Or a hedge fund could be financing the purchase of mortgage-backed securities, borrowing from his broker, for example through repo. The seller of the repurchase agreement is the borrower and sells the security used in collateral, agreeing to buy it back at a later date. In contrast, the lender buys the repurchase agreement as well as the security. (reverse-repo)

The risky asset should not be thought of too narrowly. In principle, it could represent some combination of trades, which together amount to some idiosynchratic risk exposure. At an abstract level, the "risky asset" of the paper may indeed very well be Shleifer and Vishny (1997)'s introductory example: the difference between the price of two Bund futures contracts. The belief that two Bund futures contracts, delivering the same exact value at time T, will converge before that, leads some traders to potentially take extreme positions in that direction. But they are able to maintain this position only if they have enough capital until time T, when it will converge for sure. Long-Term Capital Management was actually doing these types of trades, acting as the infinitely leveraged, very optimistic trader of the model.

No recourse. Note also that it is assumed that agents cannot impose penalties upon each other other than collateral seizure. In this model, agents do not have any income in period 1. In a model where they do have such income, or, in the case of financial intermediaries, who may have other assets, this assumption is tantamount to a no-recourse assumption.

Whether no recourse is a good assumption or not naturally depends on the nature of the risky asset, and on the identity of borrowers and lenders. In the case of a homeowner, some states in the United States are no recourse states, and the model applies very well to them. Even in recourse states, lenders do not usually go after a homeowners' personal funds after default; so the model also applies quite well to them too.

Entrepreneurs getting loans through banks usually pledge their homes or other personal assets as collateral, but these would then be included in the "equity" of the entrepreneurs, as they have the same function as personal funds. (at least to the extent that agents do not disagree about the value of these homes as well) Again, because forced servitude is not allowed in modern societies, the banker will mostly want to assess the value of the enterprise, lending on the basis of a business plan for example.

Finally, things are more subtle on financial markets, as most repurchase agreements legally allow recourse over the balance sheet of banks. However market practice suggest that lenders want their collateral to protect them fully from the risks of the investment, and do not want to worry about the creditworthiness of their counterparty. This could be because of delays, uncertainties over the deficiency judgement, or because they have no information, or do not want to know, which other trades their counterparty is engaging into. They therefore set margins as if they had no recourse on the balance sheet of their borrower.

1.3 Equilibrium

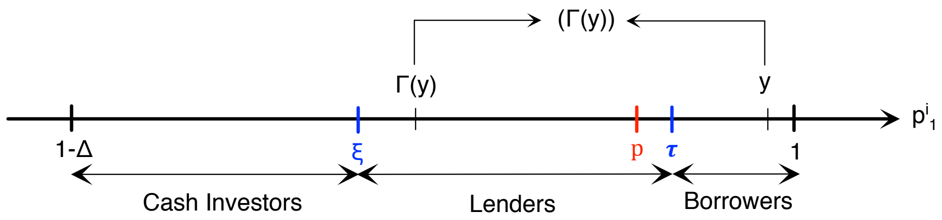
The collateral equilibrium defined in Definition 2 is characterized by two thresholds dividing the set of agents into cash investors, lenders and borrowers, a matching function $\Gamma(.)$ linking borrowers to lenders, and a bond price function $Q(.)$. The price of the asset, the two thresholds, $\Gamma(.)$, and $Q(.)$ are linked by two first-order ordinary differential equations and five algebraic equations, which can be found in the next Proposition.

Proposition 1. *(Equilibrium of Economy $\mathcal{E}_{\mathcal{B}}$) A competitive equilibrium of Economy $\mathcal{E}_{\mathcal{B}}$ is described by a price p for the Asset, two thresholds $\xi \leq \tau$, a strictly increasing matching function $\Gamma(.)$ mapping $[\tau, 1]$ onto $[\xi, \tau]$, and a loan pricing function $Q(.)$, such that:*

- *The space of agents' beliefs $[1-\Delta, 1]$ is partitioned into three intervals:*
  - *Agents with beliefs $p_1^i \in [1-\Delta, \xi]$ (cash investors) invest in Cash.*
  - *Agents with beliefs $p_1^i \in [\xi, \tau]$ (lenders) buy Borrowing Contracts.*
  - *Agents with beliefs $p_1^i \in [\tau, 1]$ (borrowers) buy the Asset and sell Borrowing Contracts.*
- *Lenders with beliefs x buy Borrowing Contracts with face value x. Borrowers with beliefs y sell Borrowing Contracts with face value $\Gamma(y)$, at price $q(\Gamma(y)) \equiv Q(y)$.*
- *p, ξ, τ, $\Gamma(.)$ and $Q(.)$ are such that:*

$$\begin{aligned} &\textit{(1a)} \quad \forall y \in [\tau, 1], \quad (p - Q(y))\, f\left(\Gamma(y)\right) \Gamma'(y) = Q(y) f(y), \\ &\textit{(1b)} \quad \forall y \in [\tau, 1], \quad (y - \Gamma(y)) Q'(y) = (p - Q(y)) \Gamma'(y). \end{aligned} \tag{1}$$

$$\begin{aligned} &\textit{(2a)} \quad 1 - F(\xi) = p, \quad \textit{(2b)} \quad \Gamma(\tau) = \xi, \quad \textit{(2c)} \quad \Gamma(1) = \tau, \\ &\textit{(2d)} \quad Q(\tau) = \xi, \quad \textit{(2e)} \quad Q(1) = \frac{\tau(p-\xi)}{\tau - \xi}. \end{aligned} \tag{2}$$



Proposition 1 is expressed with the minimum amount of formalism. I refer the reader to Appendix A.1 for a characterization of the prices of non-traded contracts, which have no impact on ex-ante or ex-post equilibrium allocations in this economy. A formal characterization of agents' portfolio choices in terms of Definition 2 is given in Appendix A.2.

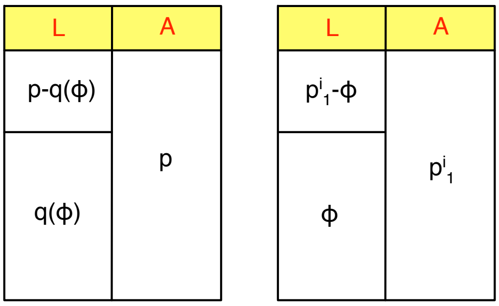
The first part of Proposition 1 establishes an intuitive result. There exists a cutoff ξ for p_1^i such that agent i with $p_1^i < \xi$ is too pessimistic to invest in the asset or lend using this asset as collateral. Agents with intermediary levels of pessimism such that $p_1^i \in [\xi, \tau]$ do not invest in the asset but lend using the asset as collateral. Finally, more optimistic agents with beliefs $p_1^i \in [\tau, 1]$ lever up into the asset.

The second and third part of the proposition both come from lenders' and borrowers' portfolio problems once these occupational choices are given. Cash investors' problem is easy: they just invest everything they have in cash. The proof then proceeds in four steps, corresponding Lemmas 1, 2, 3 and 4. I first state borrowers' problem, then lenders' problem, then the positive sorting results. I finally derive the two differential equations and five algebraic equations for the model.

Lemma 1. *(Borrowers' problem) A borrower with beliefs p_1^i chooses the face value of the Borrowing Contract ϕ to solve:*

$$\max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}.$$

The intuition for the proof is as follows. Because of the linearity of the problem, whenever an agent finds it optimal to invest in the asset, he also wants to use this asset as collateral to the maximum, in order to get the maximum amount of funds. Each time he buys a unit of the real asset, he will also sell one unit of borrowing contract (recall from Definition 1 that one borrowing contract is collateralized by one unit of the asset). The chart below illustrates the financing of a purchase of one unit of asset using a Borrowing Contract with face value ϕ, with the balance sheet in period $t = 0$ on the left-hand side and the expected balance sheet in $t = 1$ on the right-hand side:



$$\text{Return on Asset} = \frac{p_1^i}{p},$$

$$\text{Return on Debt} = \frac{\phi}{q(\phi)} = r(\phi),$$

$$\text{Return on Equity} = \frac{p_1^i - \phi}{p - q(\phi)}.$$

When he sells a contract (ϕ) with face value ϕ, he borrows $q(\phi)$ for each borrowing contract (the price of the contract). When using contract (ϕ), the borrower therefore needs to finance $p - q(\phi)$ from his own funds. The number of assets he can then buy with one unit of endowment is $1/(p - q(\phi))$ in period 0. From the expected balance sheet in period 1, his expected wealth in period 1 is then $(p_1^i - \phi)/(p - q(\phi))$. The problem of the borrower is thus to choose the contract he uses to lend to solve:

$$\max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}.$$

If the problem is interior, then Lemma 1 implies:

$$-(p - q(\phi)) + q'(\phi)(p_1^i - \phi) = 0 \quad \Rightarrow \quad \frac{p_1^i - \phi}{p - q(\phi)} = \frac{1}{q'(\phi)}.$$

This equation has a straightforward intuition: at the optimum, the benefits from choosing a higher face value for the Borrowing Contract ϕ needs to be equal to the costs. The benefit is that the borrower then can relax his borrowing constraint, and increase his equity by the amount borrowed equal to the number of Borrowing Contracts times $dq(\phi)$. The cost is that he then has to repay more in period 1, by an amount $d\phi$ times the number of Borrowing Contracts:

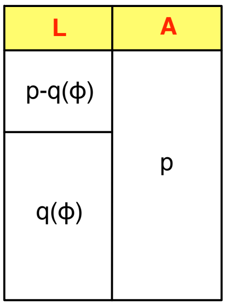
$$\underbrace{\frac{1}{p - q(\phi)}}_{\substack{\#\text{ of Borrowing}\\ \text{Contracts}}} \quad \underbrace{dq(\phi)}_{\substack{\text{Extra Funds in } t=0\\ \text{Per Unit}}} \quad \underbrace{\frac{p_1^i - \phi}{p - q(\phi)}}_{\substack{\text{Return on } t=0\\ \text{Equity}}} \quad = \quad \underbrace{\frac{1}{p - q(\phi)}}_{\substack{\#\text{ of Borrowing}\\ \text{Contracts}}} \quad \underbrace{d\phi.}_{\substack{\text{Extra Repayment}\\ \text{Per Unit}}}$$

This tradoff leads to the previous equation. For traded contracts, it must be that $q(\phi)$ increases in ϕ: no borrower would ever promise to repay more in period 1 and get less in period 0. Therefore $q(\phi)$ is increasing in ϕ. Consequently, leverage as well as $q(\phi)/(p - q(\phi))$ are increasing with ϕ. The previous expected return on equity can be written as:

$$\frac{p_1^i - \phi}{p - q(\phi)} = \left(1 + \frac{q(\phi)}{p - q(\phi)}\right)\frac{p_1^i}{p} - \frac{q(\phi)}{p - q(\phi)} r(\phi)$$

$$\frac{p_1^i - \phi}{p - q(\phi)} = \frac{p_1^i}{p} + \left(\frac{p_1^i}{p} - r(\phi)\right)\frac{q(\phi)}{p - q(\phi)}.$$

Note that different concepts are used in the financial markets to compare the amount borrowed to the value of collateral, which are all related to the price of the asset and to the amount borrowed $q(\phi)$. The chart below shows the balance sheet of a borrower for each unit of asset bought in period 0, and the definitions of leverage, loan-to-value, haircut and margin in that case.



$$\text{Leverage} = \frac{p}{p - q(\phi)} \qquad \text{Loan-To-Value} = \frac{q(\phi)}{p}$$

$$\text{Haircut} = \frac{p - q(\phi)}{p} \qquad \text{Margin} = \frac{p - q(\phi)}{q(\phi)}.$$

For the solution to this problem to be interior, it must be that there is a trade-off between leverage and return, thus $r(.)$ must also be increasing in ϕ. This brings us to lenders' problem.

Lemma 2. *(Lenders' problem) A lender with beliefs p_1^i chooses the face value of the Borrowing Contract equal to p_1^i.*

If he buys an overcollateralized contract with $\phi < p_1^i$, the lender's expected payoff is:

$$\frac{\min\{\phi, p_1^i\}}{q(\phi)} = \frac{\phi}{q(\phi)} = r(\phi),$$

which is increasing in ϕ. Hence, the lender will find it optimal to choose $\phi \geq p_1^i$. The lender does not choose an undercollateralized contract either, such that the promise exceeds the expected value of the collateral $\phi > p_1^i$. The expected payoff on an under collateralized contract is:

$$\frac{\min\{\phi, p_1^i\}}{q(\phi)} = \frac{p_1^i}{q(\phi)},$$

which is decreasing in ϕ. Therefore, it has to be that $\phi \leq p_1^i$, so $\phi = p_1^i$.

This result is intuitive. The lender does not choose an overcollateralized contract, as interest rates on loans are increasing in their face value: he is always better off lending up to the point where he thinks that the collateral will exactly cover the promised payment. He does not choose an undercollateralized contract either, as these default for sure and he needs to lend more in period 0 to get the same collateral in period 1.

Lemma 3. *(Positive sorting) More optimistic borrowers borrow with higher leverage ratio loans. They therefore effectively borrow from more optimistic lenders, through their choice of Borrowing Contracts.*

There is supermodularity of the expected wealth (and return, both being equivalent) with respect to his beliefs p_1^i and the face value of the Borrowing Contract he uses:

$$\frac{p_1^i - \phi}{p - q(\phi)} = \frac{p_1^i}{p} + \left(\frac{p_1^i}{p} - r(\phi)\right) \frac{q(\phi)}{p - q(\phi)}.$$

Therefore, the cross derivative of wealth with respect to ϕ and p_1^i is strictly positive:

$$\frac{\partial^2}{\partial p_1^i \partial \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right) = \frac{q'(\phi)}{(1 - q(\phi))^2} > 0.$$

We therefore have that:[4]

$$\begin{aligned} \frac{\partial}{\partial \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right) = 0 \quad &\Rightarrow \quad \frac{\partial^2}{\partial^2 \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right) d\phi + \frac{\partial^2}{\partial p_1^i \partial \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right) dp_1^i = 0 \\ &\Rightarrow \quad \frac{d\phi}{dp_1^i} = -\frac{\frac{\partial^2}{\partial p_1^i \partial \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right)}{\frac{\partial^2}{\partial^2 \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right)} > 0, \end{aligned}$$

from the fact that ϕ maximizes the expected return of the borrower:

$$\frac{\partial^2}{\partial^2 \phi} \left(\frac{p_1^i - \phi}{p - q(\phi)}\right) < 0$$

Therefore, the face value of the Borrowing Contract chosen by a borrower increase when he is relatively more optimistic. Since lenders choose contracts whose face value is equal to their beliefs, this means that he is also effectively borrowing from a more optimistic lender.

This result has an economic intuition. If a borrower is relatively more optimistic, he likes more buying an extra unit of the asset. Hence in the competitive equilibrium, he will be using the high leverage ratio loans, as his willingness to pay for them is higher than less optimistic borrowers. Intuitively, there is only a limited supply of optimistic lenders, and all borrowers would ideally like to borrow with high leverage ratios. But this cannot be an equilibrium, since markets would not clear. Interest rates on Borrowing Contracts thus discourage not so optimistic borrowers from using high leverage ratio loans. Denoting by $\Gamma(y)$ the face value of the Borrowing Contract chosen by a borrower with beliefs $p_1^i = y$, the positive sorting result requires that $\Gamma'(.) > 0$.

Lemma 4. *(Equations) The equilibrium quantities of the model p, ξ, and τ and $\Gamma(.)$ and $Q(.)$ for $y \in [\tau, 1]$ are given as a function of the belief cumulative distribution function $F(.)$ and its associated density $f(.)$ by two first-order ordinary differential equations (1a) and (1b), as well as five algebraic equations (2a), (2b), (2c), (2d) and (2e).*

[4] I am reluctant to refer to a planner's problem to conclude on positive sorting, as the welfare analysis of disagreement models is problematic.

Equation (1a) comes from market clearing for Borrowing Contracts with face value $\Gamma(y)$. Borrowers in an infinitesimal interval $[y, y+dy]$ of measure dy around y sell $f(y)dy/(p-Q(y))$ financial contracts with face value $\Gamma(y)$. Again, this is because for each unit asset they buy, they pay p and finance $Q(y)$ through borrowing, so they contribute $p - Q(y)$ of personal funds to the purchase of one asset. The corresponding measure of lenders dx with beliefs around $x = \Gamma(y)$ have to buy the same quantity of Borrowing Contracts, which costs $Q(y)$ each, which gives the following equation for lenders in interval $[x, x+dx]$ with wealth $f(x)dx$:

$$\underbrace{\frac{f(x)dx}{Q(y)}}_{\substack{\text{\# of Borrowing Contracts} \\ \text{bought by lenders in [x,x+dx]}}} = \underbrace{\frac{f(y)dy}{p-Q(y)}}_{\substack{\text{\# of Borrowing Contracts} \\ \text{sold by borrowers in [y,y+dy]}}}$$

$$\Rightarrow \quad \forall y \in [\tau, 1], \quad (p - Q(y))\, f(\Gamma(y))\, \Gamma'(y) = Q(y) f(y).$$

Equation (1b) results from the previously derived optimality condition for borrowers with beliefs y, resulting from borrowers' problem (see Lemma 1):

$$\frac{y-\phi}{p-q(\phi)} = \frac{1}{q'(\phi)}$$

The optimal choice of the face value for borrowers being given by $\phi = \Gamma(y)$, this writes:

$$\frac{y-\Gamma(y)}{p-q(\Gamma(y))} = \frac{1}{q'(\Gamma(y))}.$$

Since $Q(y)$ is the price at which a borrower with beliefs y sells a unit Borrowing Contract with face value $\Gamma(y)$, we have:

$$Q(y) = q(\Gamma(y)) \quad \Rightarrow \quad Q'(y) = q'(\Gamma(y))\, \Gamma'(y).$$

Therefore, replacing in the optimality condition:

$$\forall y \in [\tau, 1], \quad (y - \Gamma(y))\, Q'(y) = (p - Q(y))\, \Gamma'(y).$$

Intuitively, this second differential equation expresses the fact that interest rates rise just enough so that more pessimistic borrowers are excluded from contracts with a higher leverage ratio.

Positive sorting implies that the most optimistic borrowers sell the contracts bought by the most optimistic lenders, and symmetrically for the most pessimistic borrowers and lenders:

$$\Gamma(1) = \tau \qquad \Gamma(\tau) = \xi.$$

It must be that agents with belief $p_1^i = \xi$ are indifferent between investing in cash and lending, which pins down the return to Borrowing Contracts with the lowest leverage ratio:

$$r(\xi) = 1 \quad \Rightarrow \quad Q(\tau) = q\left(\Gamma(\tau)\right) = q(\xi) = \xi.$$

Finally, agents with belief $p_1^i = \tau$ must be indifferent between lending with the highest leverage ratio Borrowing Contracts, and investing in the asset using the lowest leverage ratio Borrowing Contracts, whose return is one and whose price is $Q(\tau) = q(\xi) = \xi$:

$$r(\tau) = \frac{\tau - \xi}{p - \xi} \quad \Rightarrow \quad Q(1) = q\left(\Gamma(1)\right) = q(\tau) = \frac{\tau}{r(\tau)} = \frac{\tau(p - \xi)}{\tau - \xi}.$$

Note finally that these equations explain why the marginal buyer of the real asset does not have beliefs equal to the price of the Real Asset, even though the reference asset has an interest rate equal to $R = 1$. The marginal buyer has beliefs $p_1^i = \tau > p$, since his outside option is to lend with a return and not to invest in Cash.

Finally, there is market clearing for the Real Asset:

$$p = 1 - F(\xi).$$

The supply of the Real Asset and wealth of each agent are both normalized to one. The left hand side is the total value of the Asset, and the right hand side is the total wealth that purchases these assets.

In sum, Lemma 4 gives an initial value problem for functions $(\Gamma(.), Q(.))$ with two first-order ordinary differential equations (1a) and (1b), as well as two initial conditions (2b) and (2d). ξ, τ and p are further determined by the remaining three algebraic equations (2a), (2c) and (2e). We are now thus ready to look at the properties of this equilibrium.

2 Equilibrium Properties

In this section, I study the properties of the equilibrium described above. I show that the shape of the upper tail of the leverage ratio distribution depends on very few characteristics of the distribution of beliefs represented by density $f(.)$. The equilibrium properties of this model depend on only two parameters: the *heterogeneity parameter* Δ (defining the support $[1-\Delta, 1]$) and the *optimism scarcity* parameter ρ, which represents the first non zero term in the Taylor expansion of $f(.)$ near 1, that is ρ is such that:

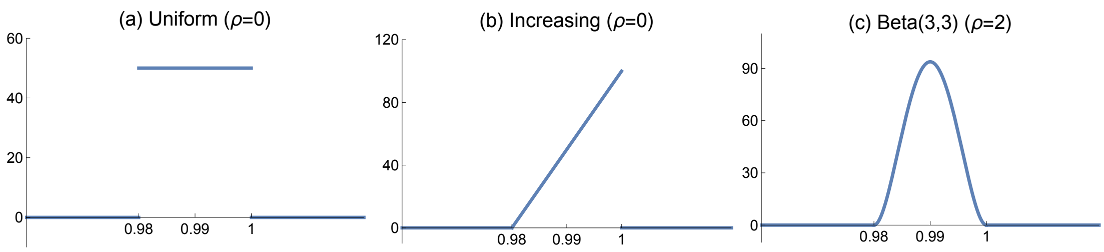
$$f\left(x\right) \sim_{x \to 1} (1 - x)^{\rho}.$$

A sufficient condition for ρ to exist is that f has a series expansion (all commonly used distributions do).[5] One can then form equivalent classes of density functions, depending on their Taylor expansions near $\max_i p_1^i = 1$, and study only the behavior of a representative of these equivalence classes, which I choose to be the density function of the Beta distribution with parameters $\rho+1$ and $\rho+1$. As a reminder, this density function as a function of the Beta function $B(.,.)$ is given by:

$$f(x) = \begin{cases} \dfrac{1}{B(\rho+1,\rho+1)} \dfrac{(x-(1-\Delta))^{\rho}(1-x)^{\rho}}{\Delta^{2\rho}} & \text{if } x \in [1-\Delta, 1] \\ 0 & \text{if } x > 1 \text{ or } x < 1-\Delta. \end{cases}$$

For example, when the density function is bounded away from zero near 1, the equilibrium is studied through the lens of the uniform distribution for beliefs, which corresponds to $\rho = 0$ in the formula above. I show using the linearly increasing density over $[1-\Delta, 1]$ as an example that the results go through as long as the density function is bounded away from zero near the maximum beliefs. As an example for what the equilibrium looks like when this assumption is violated, I use the Beta distribution Beta(3, 3) which is representative of all f that have $\rho = 2$. Those three density functions are represented below for $\Delta = 2\%$.

Figure 1: DIFFERENT DENSITIES $f(.)$ WITH $\Delta = 2\%$



Uniform distribution: closed-form solution. The model can be solved completely in closed form when the distribution is uniform, as a function of Δ. This helps build intuition on the properties of the model, for all density functions bounded away from zero near the top belief.

Lemma 5. *If the density of beliefs $f(.)$ is uniform on $[1-\Delta, 1]$, $Q(.)$, $\Gamma(.)$, p, ξ, τ, are obtained in closed form as a function of Δ:*

$$Q(\mathbf{y}) = p - p\sqrt{\left(1-\frac{Q(1)}{p}\right)^2 + 2\frac{Q(1)}{p}\left(1-\frac{Q(1)}{p}\right)\frac{1-\mathbf{y}}{1-\Gamma(1)}} \qquad \Gamma(\mathbf{y}) = \mathbf{y} - \frac{Q(\mathbf{y})}{Q'(\mathbf{y})}$$

$$\tau = \frac{1}{2} + \frac{1}{2\sqrt{1+2\Delta^2}} \qquad p = \frac{2(1+\Delta^2)}{1+\Delta+2\Delta^2+2\Delta^3+(1-\Delta)\sqrt{1+2\Delta^2}} \qquad \xi = 1-\Delta p.$$

[5] The simplest counterexample which does not satisfy this very mild regularity condition is $\exp\left(-\frac{1}{(1-x)^2}\right)$, prolonged by continuity at 1, whose derivatives in 1 are all equal to zero. It therefore goes more quickly to zero than any polynomial function of the form $(1-x)^{\rho}$.

where $Q(1)$ and $\Gamma(1)$ are given as a function of p, ξ, τ (and thus Δ) in Proposition 1.

In particular, the leverage ratio function $\frac{p}{p-Q(\mathbf{y})}$, which is one focus of the paper, can in the same way be expressed explicitly as a function of Δ only:

$$\frac{p}{p-Q(\mathbf{y})} = \frac{1}{\sqrt{\left(1-\frac{Q(1)}{p}\right)^2 + 2\frac{Q(1)}{p}\left(1-\frac{Q(1)}{p}\right)\frac{1-\mathbf{y}}{1-\Gamma(1)}}}.$$

One can perhaps already see that it is a truncated Pareto distribution with coefficient 2, since $\frac{p}{p-Q(\mathbf{y})}$ is of the form $\frac{1}{\sqrt{a+b(1-y)}}$ with a small, and y goes to 1 uniformly. I come back to this in Section 2.4. The proof for deriving Lemma 5 follows.

Proof. Equation (1a) implies when f is uniform:[6]

$$\begin{aligned}
&(p-Q)\Gamma' = Q \quad \Rightarrow \quad \Gamma' = \frac{Q}{p-Q} \quad \Rightarrow_{(1\text{b})} \quad y - \Gamma = \frac{Q}{Q'} \\
\Rightarrow \quad 1-\Gamma' = &\frac{Q'^2 - Q''Q}{Q'^2} \quad \Rightarrow_{(1\text{a})} \quad 1 - \frac{Q}{p-Q} = 1 - \frac{Q''Q}{Q'^2} \\
&\Rightarrow_{Q\neq 0} \quad Q''(p-Q) - Q'^2 = 0.
\end{aligned}$$

This is a non-linear second order differential equation in $Q(.)$, which together with initial conditions $Q(1)$ and $Q'(1)$ forms a well-defined initial value problem. Somewhat unexpectedly, this has a closed form solution as:

$$Q''(p-Q) - Q'^2 = 0 \quad \Rightarrow \quad \left(Q'(p-Q)\right)' = 0 \quad \Rightarrow \quad \left(-\frac{(p-Q)^2}{2}\right)'' = 0.$$

This together with $Q'(1) = \frac{Q(1)}{1-\Gamma(1)}$ gives $Q(y)$ in Lemma 5:

$$\begin{aligned}
-\frac{(p-Q(y))^2}{2} &= -\frac{(p-Q(1))^2}{2} + \frac{Q'(1)\,(p-Q(1))}{1-\Gamma(1)}(y-1) \\
\Rightarrow \quad Q(\mathbf{y}) &= p - p\sqrt{\left(1-\frac{Q(1)}{p}\right)^2 + 2\frac{Q(1)}{p}\left(1-\frac{Q(1)}{p}\right)\frac{1-\mathbf{y}}{1-\Gamma(1)}}
\end{aligned}$$

□

The derivation of the cutoffs is also very simple, completely analytical but a bit more cumbersome. For these, the proof is in Appendix B.

[6] I sometimes omit the dependance of functions on y in the following. For example, $Q(y)$ is denoted Q.

Other distributions: simple numerical methods. For other distributions (increasing, beta,...), I use a very straightforward numerical procedure. Assuming a ξ, τ, p, I can solve the two initial value problems defined by two ODEs (equations (1a) and (1b)) with initial conditions (equations (2b) and (2d)). The resulting final values for $Q(1)$ and $\Gamma(1)$, together with the assumed ξ, τ, p must then solve equations (2a), (2c) and (2e). The simulations shown below have been realized with Mathematica 10, which allows to solve this shooting problem very efficiently and with arbitrary precision. The code can take any density function f as input. The cutoffs are more efficiently found using a change of variable, and solving for functions $(\Gamma^{-1}, Q \circ \Gamma^{-1})$ instead of (Γ, Q). Once the cutoffs are found, (Γ, Q) are then solved for directly, in order to avoid imprecise and inefficient function inversions to compute leverage ratios.

2.1 High Amount of Financial Intermediation

When disagreement goes to zero, very few borrowers buy the real asset and effectively intermediate all the economy's funds. This is shown on Figure 3, for the case of a uniform distribution of beliefs.

[INSERT FIGURE 3 ABOUT HERE]

Note that the price of the real asset is naturally higher than it would be if no Borrowing Contracts were available (in which case, the price would simply be given by $1 - F(p) = p$). This is the leverage effect described by Geanakoplos (2003): leverage allows optimists to express themselves more, which raises asset prices.

When disagreement Δ goes to zero, Economy $\mathcal{E}_{\mathcal{B}}$ therefore does not converge to the commonly used benchmark where all agents invest their unit wealth in one unit of asset. Instead, there is complete intermediation: all agents lend to one buyer of measure zero. (one can perhaps think of these buyers as hedge funds, wealth managers, etc.) Note that there is no mathematical contradiction here: the equilibrium where one agent invests for everyone else is one of many equilibria of the setup with common beliefs. One can see the introduction of an epsilon disagreement as a refinement which selects an equilibrium, albeit one financial economists do not usually favor.

2.2 Demand for Customization, Over-The-Counter (OTC) markets

Even though my starting point was anonymous markets as in Geanakoplos (1997)' collateral equilibrium, the market equilibrium features a high degree of customization and fragmentation. It looks as if trade was done bilaterally, although the terms of trade are set perfectly competitively. This model shows that speculative markets have a natural tendency to be organized in a decentralized way, perhaps not because of search costs (although this is the way these markets are sometimes modeled), but because speculators need to find the counterparty that sets the margins they exactly need.[7] OTC markets have in fact become much more important than exchange-traded markets: according to the Bank of International Settlements, by December 2012, the OTC market was as large as $632.6 trillion globally, while the exchange-traded market was only $52.6 trillion. This model gives a new reason why trades might be OTC, namely that only two agents want to trade a specific type of contract, with a very specific margin, so that they may as well trade bilaterally.[8]

Figure 4 shows that each lender is effectively lending to a particular borrower, using one type of borrowing contract, with a specific leverage ratio (or haircut), and a specific interest rate. Borrowers with beliefs $p_1^i = y$ sell contracts $(\Gamma(y))$, which are bought by lenders with beliefs $p_1^i = \Gamma(y)$, with $\Gamma(.)$ is strictly increasing. The measure of agents buying contract $(\Gamma(y))$ is zero. Each borrower is effectively borrowing from a different lender with a different interest rate, and a different leverage ratio, both of which are higher when borrowers and lenders are relatively more optimistic. The black arrow on Figure 4 shows that, when disagreement is 10%, a lender with beliefs $p_1^i = 0.928$ is effectively lending to a borrower with beliefs 0.9984, using contract (0.928). All other agents use different contracts.

[INSERT FIGURE 4 ABOUT HERE]

The model thus may provide a framework to evaluate the costs of recent policy proposals suggesting to migrate OTC markets onto exchanges, on which investors would trade standardized contracts. This model suggests that investors may be discouraged from trading such contracts because they demand a high degree of customization, a disadvantage that could be set against the gains in terms of transparency and ease of regulation.

2.3 Hedonic Interest Rates

To the best of my knowledge, this paper is the first application of the assignment model to the study of financial markets. In the model, interest rates play the role of hedonic prices, instead of compensating lenders for expected default probability, as in textbook finance theory.

[7]Search models normally apply to markets where counterparties are hard to find, and trade is exogenously decentralized, like the labor market. According to **?** however, such is not the case of many OTC derivatives: "Certain liquid OTC derivatives (such as simple interest rate swaps and credit derivative index products), seem like natural candidates for exchange-based trade but are normally traded over the counter. At this point, we lack convincing theories that explain why such simple and heavily traded instruments are traded over the counter." In fact, even when over the counter, transactions occur among a small set of rather well identified dealer banks, and finding a counterparty is often for dealers only a matter of one or a few phone calls.

[8]To the extent that the model also applies to loans made from banks to firms, with more or less optimistic banks potentially lending to more or less optimistic firms, the sorting result can explain why matching between banks and firms appears to be non-random, as evidence from Spanish loan data in **?** suggests for example.

In the model, the implicit interest rate on Borrowing Contracts with face value (ϕ) is strictly higher than the returns to cash for all $\phi > \xi$:

$$r(\phi) = \frac{\phi}{q(\phi)} = \frac{\Gamma(y)}{Q\,(y)} > R = 1.$$

This occurs while Borrowing Contracts are fully secured according to lenders buying them, as well as borrowers selling them. This result comes from rewriting the borrowers' problem in terms of the interest rate on the Borrowing Contract $r(\phi) = \phi/q(\phi)$. One then gets:

$$\frac{y-\phi}{p-q(\phi)} = \frac{1}{q'(\phi)} \quad \Rightarrow \quad \frac{r(\phi)^2}{r(\phi)-r'(\phi)\phi} = \frac{y-\phi}{pr(\phi)-\phi}r(\phi) \quad \Rightarrow \quad \frac{r'(\phi)\phi}{r(\phi)} = \frac{y-pr(\phi)}{y-\phi}.$$

Borrowers only leverage themselves if the return on the asset is higher than the one they pay on the Borrowing Contract, so that $y/p > r(\phi)$. Moreover $y > \phi$, so that the interest rate function is strictly increasing. With the initial condition $r(\xi) = 1$ this proves that the interest rate on Borrowing Contracts is strictly higher than the returns to cash.

Depending on the level of disagreement and on the underlying distribution of beliefs, these hedonic interest rates can be of substantial magnitude. The case where f is the Beta distribution Beta$(3,3)$ (as on Figure 1c) is illustrated on Figure 5. For a standard deviation of beliefs equal to 1.89%, hedonic interest rates can be as large as 100 bps=1%. Keep in mind indeed that Δ is the difference between the maximum and minimum beliefs. For a distribution Beta$(\rho+1,\rho+1)$, the standard deviation sd is $\Delta/(2\sqrt{3+2\rho})$, so that $\Delta = 2\%$ corresponds to $sd = 0.37\%$, $\Delta = 5\%$ corresponds to $sd = 0.94\%$, and $\Delta = 10\%$ corresponds to $sd = 1.89\%$. These spreads are of the same order of magnitude as the repo rates reported in Gorton and Metrick (2012).

[INSERT FIGURE 5 ABOUT HERE]

This result on hedonic prices can potentially explain several puzzles in finance theory. For example, Bartolini et al. (2011) show that repurchase agreements' rates on Mortgage Backed Securities (MBS) secured by government-sponsored agencies, and private-label MBS are not meaningfully different from rates on unsecured interbank loans. In other words, rates on collateralized lending seem "too high".

Moreover, the idea that interest rates can be determined by something else than expected default probability is more general, and may have implications beyond the market of collateralized lending. The intuition behind hedonic interest rates is that if two borrowers compete for the same lenders, and that some of these lenders have different valuations for the asset, something else must give as both borrowers would ideally like to borrow from the lender with the highest valuation. Interest rates on bonds can thus also be market clearing prices. This insight may have applications in other financing markets. In the case of firms for example, the credit spread puzzle states that risk premia on bonds are too high to be explained by default probabilities and losses upon default. For example, Amato and Remolona (2003) show that in 1997-2003, the average spread on BBB rated corporate bonds with three to five years of maturity was about 170bps at annual rates, yet during this period the average yearly loss represented only 20bps. Collin-Dufresne et al. (2001) show that variables which should in principle determine credit spread changes have little if any explanatory power. Hedonic interest rates may provide an alternative to time varying disaster risk to explain these puzzling empirical facts.

Finally, Section 4 will provide another intuition for why rates may be determined by something else than expected default probabilities: when short-selling is an option, lenders need to be convinced to lend rather than short. Again, interest rates may be market clearing prices, and not only compensation for losses upon default.

2.4 Pareto Distributions for Leverage Ratios

The leverage $l = L(y)$ of borrower with beliefs $p_1^i = y$ is given by:

$$l = L(y) = \frac{p}{p - Q(y)}$$

Uniform Distribution. We already saw that when the belief distribution is uniform, this leverage can be written in closed form as:

$$\frac{p}{p - Q(\mathbf{y})} = \frac{p}{\sqrt{2\xi}} \sqrt{\frac{p-\xi}{\tau - \xi}} \frac{1}{\sqrt{\bar{y} - \mathbf{y}}}, \quad \text{with} \quad \bar{y} = \frac{(p+\xi)\tau - \xi(p-\xi)}{2\xi},$$

where $\bar{y}$ defined above is the belief that the most optimistic borrower would need to have so that the leverage ratio goes to infinity for the most optimistic borrowers ($y = 1$). This cutoff therefore measures what the level of truncation is. This cutoff goes to 1 as disagreement goes to zero, so that the Pareto distribution has a longer and longer upper tail (ξ, τ and p all go to one).

On Figure 6, I plot this leverage ratio distribution in the uniform case on a log-log scale, with the log in base 10 of the survivor function of leverage ratios on the y-axis, and the log in base 10 of the leverage ratio on the x-axis. That is, the log of the probability that the leverage of a borrower is higher than a certain number is plotted against the log of this number: this is a log-log or Pareto plot. Under the hypothesis that the distribution is exactly Pareto, this Pareto plot should be a straight line, whose slope is the tail index of the Pareto distribution. As can be seen on Figure 6, the distribution is not exactly Pareto. In particular, it is truncated, while the Pareto distribution has an infinite support. As said above, the level of truncation goes further out when disagreement becomes small. In practice, it is in fact well known and intuitive that empirical Pareto distributions are always truncated, which will be confirmed in Section 5 for homeowners, entrepreneurs or hedge funds. This also happens in this model, as long as disagreement is strictly positive.

[INSERT FIGURE 6 ABOUT HERE]

In may be useful at this stage to give a more direct, heuristic proof for why the leverage ratio distribution shows a Pareto distribution with a tail coefficient equal to 2, when disagreement goes to zero, which helps build intuition. Assuming away hedonic interest rates, which can be made arbitrarily small by decreasing disagreement, we would have $\Gamma(y) \sim Q(y)$ because of the definition of $r(.)$ so that $\Gamma'(y) \sim Q'(y)$. We also have then that p gets close to one. From the market clearing equation (1a) one then thus gets the approximate identity:[9]

$$(1 - Q(y))f(Q(y))Q'(y) \sim Q(y)f(y) \sim f(y) \quad \Rightarrow \quad \frac{(1 - Q(y))^2}{2} - \frac{(1 - Q(1))^2}{2} \sim 1 - y$$
$$\Rightarrow \quad 1 - Q(y) \sim \sqrt{2}\sqrt{1 - y} \quad \Rightarrow \quad L(y) \sim \frac{1}{\sqrt{2}\sqrt{1 - y}}.$$

Distributions bounded away from zero. A feature of the model is that the prediction for the Pareto coefficient is obtained irrespective of the precise shape of the function of beliefs, as long as its density function is bounded away from zero near the most optimistic beliefs. The intuition is that when f is bounded away from zero near the top beliefs, one can always approximate it by a uniform distribution ($f(y) \sim f(1)$). The above calculations are thus unchanged in the asymptotic case where y goes to 1 and disagreement is small. Figure 7 shows the example of the increasing distribution over $[1 - \Delta, 1]$. Experimenting with different f shows that the range on which the Pareto approximation is good is very large as long as f is not too fast moving.

[INSERT FIGURE 7 ABOUT HERE]

There is actually a clear intuition for why the distribution of leverage ratios converges to that of an asymptotic Pareto distribution of tail coefficient equal to two, it is worth giving first an intuition for it. The leverage ratio, in the space of lenders' beliefs, is proportional to a ratio of one over the difference between the collateral price and the price of the Borrowing Contract, which goes to zero when more optimistic agents are lending. The reason for why margins go to zero is that the most optimistic lenders would almost be willing to buy the asset at the going price. When the distribution of beliefs is sufficiently regular, this difference can be approximated by a uniform distribution in the limit, because it is close to a difference between lenders' beliefs and a constant (up to an interest rate term though, which is negligible in the limit, as above). The reason

[9] One can alternatively integrate directly without the last approximation $Q(y)f(y) \sim f(y)$, and then do a Taylor expansion of the logarithm near 1.

why the exponent of this Pareto is two, and not one, is that what is measured is the leverage ratio in borrowers' space, not in lenders' space. Because of the properties of the matching function $\Gamma(.)$, many lenders are matched with very few borrowers asking for close to zero margins, and therefore the measure of the corresponding lenders is higher than the measure of borrowers. Going from lenders' space to borrowers' space then involves a square transformation. Therefore, the distribution has a shape exponent of exactly two as long as the density function of beliefs is bounded away from zero.

Distributions non bounded away from zero. In the case where the distribution is not bounded away from zero, and if the optimism scarcity parameter ρ exists (see above), then the leverage ratio distribution is Pareto with a coefficient equal to $2+\rho$ for small disagreement. The case where $f = \text{Beta}(3,3)$ is illustrated on Figure 8. Again, all density functions with $\rho = 2$ have the same Pareto coefficient for small disagreement.

[INSERT FIGURE 8 ABOUT HERE]

The simplest way to get this result is to solve numerically for the system of ordinary differential equations (1a) and (1b). One can also get this result with the same heuristic method as above:

$$\begin{aligned}
&(1-Q(y))f(Q(y))Q'(y) \sim f(y) \quad \Rightarrow \quad (1-Q(y))^{\rho+1}\, Q'(y) \sim (1-y)^{\rho} \\
&\Rightarrow \quad \frac{(1-Q(y))^{\rho+2}}{\rho+2} \sim \frac{(1-y)^{\rho+1}}{\rho+1} \quad \Rightarrow \quad 1-Q(y) \sim \left(\frac{\rho+2}{\rho+1}\right)^{\frac{1}{\rho+2}} (1-y)^{\frac{\rho+1}{\rho+2}} \\
&\Rightarrow \quad L(y) \sim \frac{\left(\frac{\rho+1}{\rho+2}\right)^{\frac{1}{\rho+2}}}{(1-y)^{\frac{\rho+1}{\rho+2}}} \quad \Rightarrow \quad L(y) \sim \frac{A}{(1-F(y))^{\frac{1}{\rho+2}}},
\end{aligned}$$

for some $A > 0$. This is a Pareto distribution with coefficient $\rho + 2$ in the space of borrowers. Note that for $\rho = 0$, one recovers the case of functions that are bounded away from zero near the top beliefs, with a Pareto coefficient equal to 2.

2.5 Expected Returns of Borrowers

- As was shown previously, the expected return of a borrower with beliefs y is given by:

$$\frac{y-\Gamma(y)}{p-Q(y)} = \underbrace{\frac{y}{p}}_{\substack{\text{Returns from}\\ \text{Own Equity}}} + \underbrace{\left(\frac{p}{p-Q(y)}-1\right)}_{\text{Leverage}-1} \underbrace{\left(\frac{y}{p}-\frac{\Gamma(y)}{Q(y)}\right)}_{\substack{\text{Returns from Leveraging}\\ \text{for One Unit}}}.$$

There are two key components of this expected return. On the one hand, optimistic borrowers tend to have a very high leverage, as shown before. On the other hand, for markets to clear, borrowers give some part of the surplus they expect to gain on each unit of the asset they buy to the lender they borrow from. Moreover, the more they bid up the price of the asset, the lower their expected return on each unit of asset, which again drives down the returns from leveraging into one unit of asset. It is not clear a priori which of these forces should dominate.

Uniform Distribution. With a uniform distribution, borrowers compete a lot for the highest leverage ratio loans, so that they effectively give all their expected returns to lenders. The unit return is thus going to zero for the most optimistic borrowers. As shown on Figure 9, this undoes the leverage effect. The expected returns of borrowers are of the order of magnitude of the underlying disagreement on the real asset: there is no amplification through leverage.

[INSERT FIGURE 9 ABOUT HERE]

$\rho > \mathbf{1}$. Things are different when $\rho > 1$. Again, the case of $f = \text{Beta}(3,3)$ (then $\rho = 2$) is illustrated on Figure 10. Because the most optimistic borrowers are in relatively scarce supply, they compete less for the most optimistic lenders. While the magnitude of the leverage ratio they achieve is still very high, they now do not need to transfer all their expected surplus from buying assets to lenders through hedonic interest rates.

[INSERT FIGURE 10 ABOUT HERE]

The distribution of these expected returns is then close to being Pareto, as illustrated on Figure 11 with $f = \text{Beta}(3,3)$. One can at least see that these returns are highly heterogenous, fat tailed, and skewed.[10]

[INSERT FIGURE 11 ABOUT HERE]

The intuition for why these expected returns are close to Pareto is that they come from the integration of the leverage ratio distribution, which is itself a truncated Pareto. Indeed, from the envelope condition, we have that expected returns of borrowers are such that:

$$R'(y) = \frac{1}{p - Q(y)} = \frac{L(y)}{p}.$$

Thus, the Pareto property for $L(y)$ translates into a high variance, fat tailed and skewed expected return function.

[10] The fact that their variance is very high, means that the mean of the returns does not go to a Gaussian distributions, but to a Pareto-Levy stable distribution.

2.6 Non Monotonic Relationship between Leverage and Realized Returns

Conditional on a price realization for the risky asset p_1 in period 1, the agent making the highest return is intuitively the one who was entertaining that belief at time 0. For borrowers, there is therefore a non-monotonic relationship between leverage and ex-post realized returns. Even when $p_1 > p$ (that is, the risky asset has a higher return than cash ex-post), the realized return of a borrower is non monotonic as a function of his beliefs, and therefore his leverage, whenever the price of realization p_1 of the risky asset is interior ($p_1 < 1$):

$$\frac{\partial}{\partial y}\left(\frac{p_1 - \Gamma(y)}{p - Q(y)}\right) = \frac{-\Gamma'(y)\,(p - Q(y)) + Q'(y)(p_1 - \Gamma(y))}{(p - Q(y))^2}$$

Using borrowers' optimality condition $(y - \Gamma)Q' = (p - Q)\Gamma'$, this gives:

$$\frac{\partial}{\partial y}\left(\frac{p_1 - \Gamma(y)}{p - Q(y)}\right) = \frac{Q'(y)(p_1 - y)}{(p - Q(y))^2}.$$

Therefore the derivative of borrowers' realized returns has the sign of $p_1 - y$, which shows a non-monotonic relationship between leverage and returns. Borrowers' realized return increases with leverage if they were too pessimistic relative to the truth but it decreases with leverage if they were too optimistic.

The intuition comes once again from the hedonic interest rates. It is not sufficient for a borrower to know whether the asset will yield positive excess returns or not, but by how much. Figure 12 illustrates the relationship between leverage and returns depending on realized p_1, and contrasts them with ex-ante subjective returns, in the case where $f = \text{Beta}(3,3)$ and $\Delta = 2\%$.

[INSERT FIGURE 12 ABOUT HERE]

In textbook finance theory, where agents are risk averse, there is in contrary a positive relationship between leverage and returns. Higher leverage into the market portfolio leads to a higher covariance with the market risk (higher β), and thus must be compensated by a higher realized return on average. Implicitly, borrowing and lending is done unconstrained at the risk free rate. The present model suggests that disagreement can confound the clear theoretical link between leverage and returns, and thus may lead to a failure of the CAPM.

Finally, it should be noted that when the realization p_1 is higher than what the most optimistic agent was thinking, the return realization is a truncated Pareto distribution, and inherits exactly the shape of the leverage ratio distribution.

3 Pyramiding / Tranching Economies

In the Borrowing Economy $\mathcal{E}_{\mathcal{B}}$, I exogenously imposed that agents could only use the real asset as collateral. In practical terms, this happens when loans remain on banks' balance sheets without being securitized. But a Borrowing Contract is also an asset, albeit a financial asset, available in zero net supply, which can be transformed into a security. When they are, they can also be used to back promises.

3.1 Setup

In this section, I work in the same setting as in Economy $\mathcal{E}_{\mathcal{B}}$ (see Section 1.1), except that I allow agents to agree to collateralized Borrowing Contracts not only using the Real Asset as collateral, but also using Borrowing Contracts collateralized by the Real Asset as collateral, which I refer to as *Borrowing Contracts Squared*. I call this Economy the Pyramiding or Tranching Economy $\mathcal{E}_{\mathcal{B}^2}$. Formally,

Definition 3. (Borrowing Contract Squared, Economy $\mathcal{E}_{\mathcal{B}^2}$) A *Borrowing Contract Squared* $(\phi')^{(2)}$ in Economy $\mathcal{E}_{\mathcal{B}^2}$ is a promise of $\phi' \geq 0$ units of Cash in period 1, the *face value*, collateralized by one unit of Borrowing Contract (ϕ).

Note that restricting ourselves to $\phi' \leq \phi$ is without loss of generality, as Borrowing Contracts Squared with $\phi < \phi'$ have the same payoffs as Borrowing Contracts (ϕ'), no matter what p_1 is. The reason is that the holder of the Borrowing Contract always wants to default for $\phi < p_1 < \phi'$. Note also that the face value ϕ of the Borrowing Contract backing the Borrowing Contract Squared $(\phi')^{(2)}$ is irrelevant, as long as $\phi' \leq \phi$. The reason is that the collateral matters only for price realization such that $p_1 \leq \phi'$ and that for those price realizations, the corresponding Borrowing Contract also is in default, such that only the collateral matters. This is why the *Borrowing Contract Squared* $(\phi')^{(2)}$ can be referred to by its face value only. The price of this Borrowing Contract Squared will be denoted by $q_2(\phi')$. On the other hand, regular Borrowing Contracts are now traded at competitive price $q_1(\phi)$ when of face value ϕ (to distinguish their price from that of Borrowing Contracts Squared).

Equilibrium. Again, all units of the real asset are initially endowed to unmodeled agents who sell their asset holdings and then consume. Agent i chooses his position in the Real Asset n_A^i, a menu of financial Borrowing Contracts (ϕ) denoted by $dN_B^i(.)$, a menu of Borrowing Contracts Squared denoted by $dN_{B^2}^i(.)$, and Cash n_C^i, in order to maximize his expected wealth in period 1 according to his subjective beliefs p_1^i about the Real Asset (W), subject to his budget constraint (BC), a collateral constraint for Borrowing Contracts (CC) and a second collateral constraint for Borrowing Contracts Squared (CC2):

$$\max_{(n_A^i, dN_B^i(.), dN_{B^2}^i(.), n_C^i)} n_A^i p_1^i + \int_\phi \min\{\phi, p_1^i\} dN_B^i(.) + n_C^i + \int_{\phi'} \min\{\phi', p_1^i\} dN_{B^2}^i(\phi') \tag{W}$$

$$\text{s.t.} \quad n_A^i p + \int_\phi q_1(\phi) dN_B^i(\phi) + n_C^i + \int_{\phi'} q_2(\phi') dN_{B^2}^i(\phi') \leq 1 \tag{BC}$$

$$\text{s.t.} \quad \int_\phi \max\{-dN_B^i(\phi), 0\} d\phi \leq n_A^i \tag{CC}$$

$$\text{s.t.} \quad \int_{\phi'} \max\{dN_{B^2}^i(\phi'), 0\} \leq \int_\phi dN_B^i(\phi) \tag{CC2}$$

$$\text{s.t.} \quad n_A^i \geq 0, \quad n_C^i \geq 0$$

Note that each time a borrower/lender sells of unit of Borrowing Contract Squared, he needs to own one unit of Borrowing Contract, which explains equation (CC2). A *Competitive Equilibrium* for Economy $\mathcal{E}_{\mathcal{B}^2}$ is then a price p for the Real Asset, a distribution of prices $q_1(.)$ for all traded Borrowing Contracts (ϕ), of prices $q_2(.)$ for all traded Borrowing Contracts Squared $(\phi')^{(2)}$ and portfolios $(n_A^i, dN_B^i(.), dN_{B^2}^i(.), n_C^i)$ for all agents i in the Real Asset, Borrowing Contracts, Borrowing Contracts Squared and Cash, such that all agents i maximize expected wealth (W) according to their subjective beliefs, subject to their budget constraint (BC), the collateral constraints (CC) and (CC2), and markets for the Real Asset Borrowing Contracts and Borrowing Contracts Squared clear:

$$\int_i n_A^i di = 1, \tag{MC_A}$$

$$\forall \phi, \quad \int_i dN_B^i(\phi) di = 0. \tag{MC_B}$$

$$\text{and} \quad \forall \phi', \quad \int_i dN_{B^2}^i(\phi') di = 0. \tag{MC_{B^2}}$$

3.2 Equilibrium

The equilibrium of the Pyramiding Economy is now characterized by three scalars dividing the set of agents into cash investors, Lenders of Type-2 (lending to lenders), Lenders of Type 1 (lending to ultimate borrowers), and borrowers, one scalar for the price of the real asset, a function for the price all traded Borrowing Contracts, another for the price of all traded Borrowing Contracts Squared, a function determining the optimal choice of borrowers' contracts, and another determining the optimal choice of lenders. These equilibrium objects are linked by four first-order ordinary differential equations and eight algebraic equations, given in the next proposition.

Proposition 2. *(Equilibrium of Economy $\mathcal{E}_{\mathcal{B}^2}$) A competitive equilibrium of Economy*

$\mathcal{E}_{\mathcal{B}^2}$ is described by a price p for the Asset, three thresholds $\nu \leq \xi \leq \tau$, two strictly increasing matching functions $\Gamma_1(.)$ mapping $[\tau, 1]$ onto $[\xi, \tau]$ and $\Gamma_2(.)$ mapping $[\xi, \tau]$ onto $[\nu, \xi]$, and two loan pricing functions $Q_1(.)$ and $Q_2(.)$, such that:

The space of agents' beliefs $[1-\Delta, 1]$ is partitioned into four intervals:

Proposition 3. *Agents i with beliefs $p_1^i \in [1-\Delta, \nu]$ (cash investors) invest in Cash.*

Agents i with beliefs $p_1^i \in [\nu, \xi]$ (lenders of Type-2) buy Borrowing Contracts Squared.

Agents i with beliefs $p_1^i \in [\xi, \tau]$ (lenders of Type-1) buy Borrowing Contracts and sell Borrowing Contracts Squared.

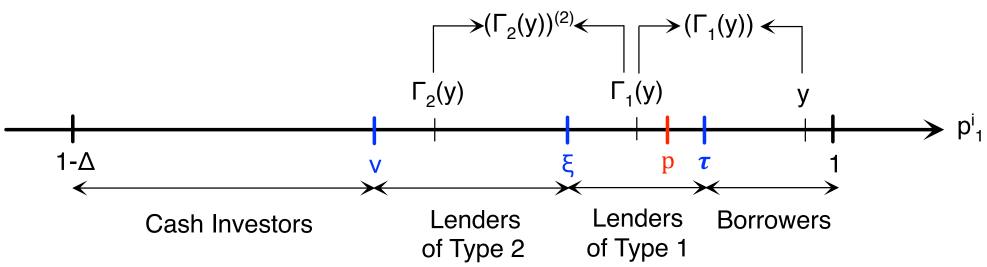
Agents i with beliefs $p_1^i \in [\tau, 1]$ (borrowers) buy the Asset and sell Borrowing Contracts.

Lenders of Type-1 with beliefs x buy Borrowing Contracts with face value x. Borrowers with beliefs y sell Borrowing Contracts with face value $\Gamma_1(y)$, at price $Q_1(y)$. Lenders of type 2 with beliefs z buy Borrowing Contracts Squared with face value z. Lenders of Type-1 with beliefs $\Gamma_1(y)$ sell Borrowing Contracts Squared with face value $\Gamma_2(y)$, at price $Q_2(y)$.

p, ν, ξ, τ, $\Gamma_1(.)$, $\Gamma_2(.)$, $Q_1(.)$, and $Q_2(.)$ are such that:

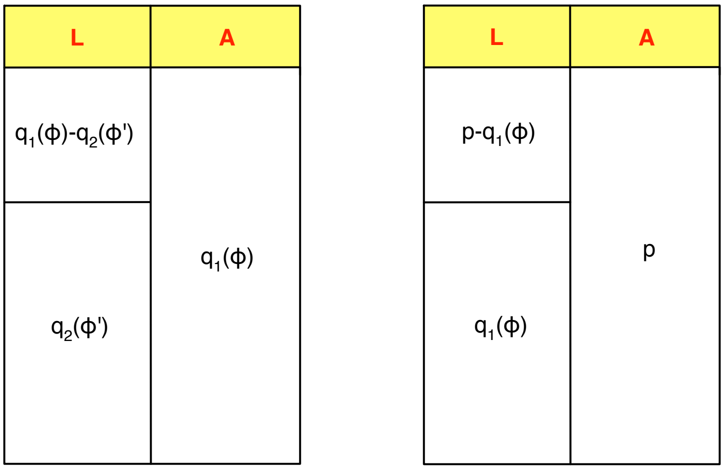
$$\begin{aligned} &\textit{(3a)} \quad (p - Q_1) f(\Gamma_2) \Gamma_2' = Q_2 f, \qquad &&\textit{(3b)} \quad (p - Q_1) f(\Gamma_1) \Gamma_1' = (Q_1 - Q_2) f \\ &\textit{(3c)} \quad (y - \Gamma_1) Q_1' = (p - Q_1) \Gamma_1', \qquad &&\textit{(3d)} \quad (\Gamma_1 - \Gamma_2) Q_2' = (Q_1 - Q_2) \Gamma_2'. \end{aligned} \tag{3}$$

$$\begin{aligned} &\textit{(4a)} \quad 1 - F(\nu) = p, \\ &\textit{(4b)} \quad \Gamma_1(\tau) = \xi, \quad \textit{(4c)} \quad \Gamma_1(1) = \tau, \quad \textit{(4d)} \quad \Gamma_2(\tau) = \nu \quad \textit{(4e)} \quad \Gamma_2(1) = \xi \\ &\textit{(4f)} \quad Q_2(\tau) = \nu, \quad \textit{(4g)} \quad \frac{\xi}{Q_2(1)} = \frac{\xi - \nu}{Q_1(\tau) - \nu} \quad \textit{(4h)} \quad \frac{\tau - \xi}{Q_1(1) - Q_2(1)} = \frac{\tau - \xi}{p - Q_1(\tau)}. \end{aligned} \tag{4}$$



The fact that $\Gamma_2(.)$ does not match the beliefs of a lender of type-1 to a that of a lender of type-2 might be a bit confusing at first, but it actually allows for more simple expressions. The reason is that borrowers are of particular interest here, and so it is simpler to calculate all functions taking their beliefs as a reference. It is also in this way that the equilibrium needs to be computed numerically.

Figure 2: PYRAMIDING ARRANGEMENT, TRANCHING OF COLLATERAL



Note: The borrowers' balance sheet is shown on the right hand side: he finances the purchase of the asset using this asset as collateral with a lender of type 1, financing $q_1(\phi)$ towards the purchase. This lender himself gets financing from lenders of type 2, who contribute $q_2(\phi')$ towards the purchase of the collateral. For example, an investor buying a Mortgage-Backed Security lends to a homeowner; this investor finances part of his position through using the Mortgage-Backed Security in repo. Another example is a hedge fund who posts collateral with his broker dealer. The broker dealer rehypothecates the collateral to finance part of the position, with a money market fund. A final example is the **tranching** of loans. Lender of type-2 would have a senior tranche, and be repaid in full whenever the value of the collateral exceeds ϕ'. Lender of type 1 would have a junior tranche, and will start being repaid only if the value of the collateral exceeds ϕ', and be repaid in full if it exceeds ϕ.

A full proof of this proposition would exactly mirror that of Proposition 1. I shall only focus on the minor changes brought about by the presence of pyramiding arrangements. Obviously, the main difference is that Borrowing Contracts can now be used by Lenders of Type-1 to finance themselves from even more pessimistic agents. The intuition for why they want to do that in equilibrium is that borrowers give them part of their returns because of the allocative argument, and that Lenders of Type-1 therefore want to purchase as many of these contracts as possible, possibly using Borrowing Contracts themselves as collateral. This is illustrated on Figure 2.

As for the Borrowing Economy, a first equation results from the market clearing for Borrowing Contracts Squared with face value $\Gamma_2(y) = z$. These contracts are bought by lenders of type 2 in a small interval $[z, z + dz]$, with wealth $f(z)dz$, and sold by lenders of type 1 in a small interval $[x, x + dx]$. Lenders of type 2 therefore contribute $Q_2(y)$ to the total loan amount that lenders of type 1 make to borrowers, and lenders of type 1 contribute $Q_1(y) - Q_2(y)$ from their own funds. (see the balance sheet on Figure 2)

$$\underbrace{\frac{f(z)dz}{Q_2(y)}}_{\substack{\#\text{ of Borrowing Contracts Squared}\\ \text{bought by lenders of Type 2 in [z,z+dz]}}} = \underbrace{\frac{f(x)dx}{Q_1(y) - Q_2(y)}}_{\substack{\#\text{ of Borrowing Contracts Squared}\\ \text{sold by lenders of type 1 in [x,x+dx]}}}$$

$$\Rightarrow \quad \forall y \in [\tau, 1], \quad (Q_1(y) - Q_2(y)) \, f(\Gamma_2(y)) \, \Gamma_2'(y) = Q_2(y) f(\Gamma_1(y)) \, \Gamma_1'(y).$$

Moreover, one can write the market clearing for Borrowing Contracts with face value $\Gamma_1(y) = x$. These contracts are bought by lenders of type 1 (using their own funds and lenders of type 2's funds, as seen above) in a small interval $[x, x + dx]$, so that:

$$\underbrace{\frac{f(x)dx}{Q_1(y) - Q_2(y)}}_{\substack{\text{\# of Borrowing Contracts} \\ \text{bought by lenders of Type 1 in [x,x+dx]}}} = \underbrace{\frac{f(y)dy}{p - Q_1(y)}}_{\substack{\text{\# of Borrowing Contracts} \\ \text{sold by borrowers in [y,y+dy]}}}$$

$$\Rightarrow \quad \forall y \in [\tau, 1], \quad (p - Q_1(y)) \, f(\Gamma_1(y)) \, \Gamma_1'(y) = (Q_1(y) - Q_2(y)) \, f(y).$$

Multiplying the two sides of the above equations, and keeping the second equation, allows to conclude that the allocation functions are given by the following differential equations:

$$\begin{aligned} (p - Q_1(y)) \, f(\Gamma_2(y)) \, \Gamma_2'(y) &= Q_2(y) f(y) \\ (p - Q_1(y)) \, f(\Gamma_1(y)) \, \Gamma_1'(y) &= (Q_1(y) - Q_2(y)) \, f(y) \end{aligned}$$

The choice of Borrowers with beliefs y regarding the face value of Borrowing Contracts, gives a first differential equation:

$$\begin{aligned} \max_\phi \frac{y - \phi}{p - q_1(\phi)} \quad &\Rightarrow \quad -(p - q_1(\phi)) + q_1'(\phi)(y - \phi) = 0 \\ &\Rightarrow \quad (p - Q_1(y)) \, \Gamma_1'(y) = (y - \Gamma_1(y)) \, Q_1'(y). \end{aligned}$$

The choice of Lenders of Type 1, with beliefs x_1, and who choose Borrowing Contracts Squared to lever themselves into the spreads given by traditional Borrowing Contracts, gives a second differential equation:

$$\begin{aligned} \max_\phi \frac{x_1 - \phi}{q_1(x_1) - q_2(\phi)} \quad &\Rightarrow \quad -(q_1(x_1) - q_2(\phi)) + q_2'(\phi)(x_1 - \phi) = 0 \\ &\Rightarrow \quad (Q_1(y) - Q_2(y)) \, \Gamma_2'(y) = (\Gamma_1(y) - \Gamma_2(y)) \, Q_2'(y). \end{aligned}$$

Just as previously, the market clearing equation for the real asset writes:

$$1 - F(\nu) = p.$$

This is because now the funds of Lenders of type 2 are also invested in the asset. Positive sorting at the boundaries brings:

$$\Gamma_1(\tau) = \xi, \quad \Gamma_1(1) = \tau, \quad \Gamma_2(\tau) = \nu, \quad \Gamma_2(1) = \xi.$$

Finally, indifference for agents with beliefs ν, ξ, and τ respectively imply:

$$\begin{aligned}
1 = \frac{\nu}{q_2(\nu)} \quad &\Rightarrow \quad Q_2(\tau) = \nu \\
\frac{\xi}{q_2(\xi)} = \frac{\xi-\nu}{q_1(\xi)-q_2(\nu)} \quad &\Rightarrow \quad \frac{\xi}{Q_2(1)} = \frac{\xi-\nu}{Q_1(\tau)-Q_2(\tau)} \quad \Rightarrow \quad \frac{\xi}{Q_2(1)} = \frac{\xi-\nu}{Q_1(\tau)-\nu} \\
\frac{\tau-\xi}{q_1(\tau)-q_2(\xi)} = \frac{\tau-\xi}{p-q_1(\xi)} \quad &\Rightarrow \quad \frac{\tau-\xi}{Q_1(1)-Q_2(1)} = \frac{\tau-\xi}{p-Q_1(\tau)}.
\end{aligned}$$

3.3 Discussion

Equivalence to tranching. Note that the above decomposition of claims between Borrowing Contracts and Borrowing Contracts Squared has an equivalent representation in terms of a junior and a senior tranche from a securitized mortgage. On Figure 2, Lender of Type-2 gets repaid whenever $p_1 \geq \phi'$, and lender of Type-1 is junior as he gets repaid in full only if $p_1 \geq \phi$.

Importance of pyramiding. Borrowing Contracts Squared can have several real world interpretations. For example, prime brokers to hedge funds are often allowed to use their clients' collateral for financing from money market funds (money market funds usually are not allowed to lend to hedge funds directly by regulation). Another round of lending occurs when Mortgage-Backed Securities are financed through repurchase agreements. Greenwood and Scharfstein (2013) calculate that fixed income securities grew from totaling 57% of GDP in 1980 to 182% of GDP in 2007, and 58% of the growth of fixed income securities came from securitization.

Limits to pyramiding. The assumption made in Section 1 is still a good description for some markets. One reason is regulation. In the United States for example, Regulation T and SEC's Rule 15c3 limit Prime Brokers' use of rehypothecated collateral from a client. Similarly, securitization in the housing market is a fairly recent phenomenon, the extent of which was broadened in the nineties in the United States. However, it is still not possible in many countries around the world for regulatory reasons. Apart from institutional and regulatory obstacles, securitization may entail some real cost (again, see Greenwood and Scharfstein (2013)). The higher disagreement, the more these costs are likely to be outweighed by expected gains.

More layers of pyramiding? For simplicity, I have investigated only the case where Borrowing Contracts can be used as collateral to get more funds. Naturally, there is no reason why there should be exactly two layers of lending in all circumstances. The model straightforwardly generalizes to a case with an arbitrary number of pyramids of lending.[11] A real world example would be the tranching of a mortgage into six different levels of riskiness (AAA, AA,...). Formally, this would correspond to Economy $\mathcal{E}_{\mathcal{B}}{}^{6}$, with a six levels "waterfall".

3.4 Results

In this section, I present the results obtained from the equilibrium in the Pyramiding Economy (Proposition 2), and compare them to that of the equilibrium in the Borrowing Economy.

More bias towards optimists. Importantly, I show that pyramiding lending arrangements are not redundant to the use of leverage by borrowers. The reason is that it allows lenders to also obtain more funds from less optimistic lenders. Just as leverage increases asset prices and allows optimists to express their opinions more, pyramiding, that is lending using Borrowing Contracts as collateral (or tranching), also increases the price of the real asset. Figure 13 compared to Figure 3 shows that the price of the real asset is considerably increased by the availability of pyramiding lending arrangements.

[INSERT FIGURE 13 ABOUT HERE]

Leverage ratio distribution. Leverage ratios of borrowers increase dramatically in the Pyramiding Economy, as can be seen in the case where $f = \text{Beta}(3,3)$ and $\Delta = 2\%$ on Figure 14. Leverage ratios of final borrowers are given as follows in the Borrowing and the Pyramiding Economy respectively.

$$\text{Borrowing:} \quad \frac{p}{p - Q(y)} \qquad \text{Pyramiding:} \quad \frac{p}{p - Q_1(y)}.$$

[INSERT FIGURE 14 ABOUT HERE]

The intuition is that borrowers' final leverage results both from their borrowing from Lenders of Type-1, but also that their direct lenders also leverage themselves from lenders of type-2.

As can be seen on Figure 15, this multiplication of two leverage ratio distributions leads to a decrease in the Pareto coefficient, as the leverage ratio distribution becomes more fat tailed. In fact, it is even possible to get a formula for the tail coefficient in closed form, just as in the case of the Borrowing Economy. For example, when the density function is bounded away from zero near the top skills, one can show that the Pareto coefficient decreases from 2 to 3/2. The decrease in the Pareto coefficient is more general, whatever ρ. One can solve for the model numerically to arrive at this result. One can also use the same heuristic methods as before, approximating the market

[11] The number of these layers can be pinned down by adding a small cost of adding a layer of pyramiding, as the gains from adding another layer are decreasing in the number of layers.

clearing equations (3a) and (3b), for low disagreement. One can use both equations to arrive at an approximation for the leverage of Type-1 lenders:

$$(Q_1(y) - Q_2(y))\, f\,(\Gamma_2(y))\, \Gamma_2'(y) = Q_2(y) f\,(\Gamma_1(y))\, \Gamma_1'(y)$$
$$\Rightarrow \quad \frac{(1 - Q_1(y))^{\rho+1}}{\rho+1} \sim \frac{(1 - Q_2(y))^{\rho+2}}{\rho+2}.$$

One then uses this approximation to plug into the market clearing condition giving final borrowers' leverage:

$$(p - Q_1(y))\, f\,(\Gamma_1(y))\, \Gamma_1'(y) = (Q_1(y) - Q_2(y))\, f(y)$$
$$\Rightarrow \quad (1 - Q_1(y))^{\rho+1}\, Q_1'(y) \sim (1 - Q_2(y))\, f(y)$$
$$\Rightarrow \quad (1 - Q_1(y))^{\rho+1}\, Q_1'(y) \sim A\,(1 - Q_1(y))^{\frac{\rho+1}{\rho+2}}\,(1-y)^{\rho}$$
$$\Rightarrow \quad 1 - Q_1(y) \sim (1-y)^{\frac{\rho+1}{\frac{(\rho+1)^2}{\rho+2}+1}}$$
$$\Rightarrow \quad L(y) \sim \frac{1}{(1-y)^{\frac{\rho+1}{\frac{(\rho+1)^2}{\rho+2}+1}}} \quad \Rightarrow \quad L(y) \sim \frac{1}{(1-F(y))^{\frac{1}{\frac{(\rho+1)^2}{\rho+2}+1}}}.$$

For $\rho = 0$, we indeed obtain a tail coefficient equal to $3/2$. One can then analytically compare the corresponding tail exponents on the leverage ratio distributions for all ρ, and show that the tail coefficient is lower in the Pyramiding Economy than in the Borrowing Economy since:

$$\frac{(\rho+1)^2}{\rho+2} + 1 < \rho + 2 \quad \Leftrightarrow \quad \rho > -1.$$

The numerical simulation on Figure 14 shows an example of this result.

In Section 5.1, it is shown that this characterization of pyramiding lending arrangements in terms of their effect on the tail coefficient of the leverage ratio distribution could have alerted regulators in real time of mounting risks in the financial system.

Expected returns of borrowers. Expected returns of borrowers also increase a lot, as can be seen on Figure 15, which shows the case where $f = \text{Beta}(3,3)$ again. Expected returns by borrowers in the Borrowing and Pyramiding economies are defined as follows.

$$\text{Borrowing:} \quad \frac{y - \Gamma(y)}{p - Q(y)} \qquad \text{Pyramiding:} \quad \frac{y - \Gamma_1(y)}{p - Q_1(y)}.$$

[INSERT FIGURE 15 ABOUT HERE]

Expected returns of lenders. Finally, expected returns of lenders obviously rise in the pyramiding economy, as lenders are able to lever into hedonic interest rates.

Figure 16 shows the results when $f = \text{Beta}(3,3)$ and $\Delta = 2\%$: lenders can achieve returns of 6% in the pyramiding economy, while they were getting less than 20 basis points in the Borrowing Economy. Fixed income thus can generate considerable revenue, provided there is pyramiding or tranching. Note also on this graph that direct lenders to borrowers are also relatively more optimistic in the pyramiding economy, which helps explain why the leverage of ultimate borrowers is so high. Expected returns in the Borrowing and Pyramiding economies are defined as follows.

$$\text{Borrowing:} \quad \frac{\Gamma(y)}{Q(y)} \qquad \text{Pyramiding:} \quad \frac{\Gamma_1(y) - \Gamma_2(y)}{Q_1(y) - Q_2(y)}.$$

[INSERT FIGURE 16 ABOUT HERE]

Importantly, it is not that pyramiding completes market in the sense that new contingent assets are introduced. The effect of pyramiding is to allow the same collateral to back two different promises at the same time. Lenders of type-2 are in equilibrium using Borrowing Contracts Squared, which are equivalent (in terms of payoffs) to regular Borrowing Contracts with the same face value. However, they were not using them in equilibrium of Economy $\mathcal{E}_{\mathcal{B}}$ for lack of available collateral. To the best of my knowledge, it is the first time that what Geanakoplos (1997) calls *pyramiding lending arrangements* or *double leverage cycle* are being modeled.

4 Extension: Borrowing with Short-Sales Economy

The above analysis was carried out with a maintained assumption of no short-sales constraints, in the tradition of Miller (1977). This is a good approximation for some markets, for example if it concerns housing or a new entrepreneurial investment. However in many markets today (large stocks are an example), going short is almost as easy as going long. The framework I have developed above extends very naturally to the case where such short-selling is possible. I now study such a Borrowing with Short-Sales Economy $\mathcal{E}_{\mathcal{S}}$, where both borrowing and short-selling are limited by the endogenous availability of collateral.

Simsek (2013) was, to the best of my knowledge, the first to introduce endogenous short-sales constraints in a disagreement model. However the scope of his analysis is limited by his assumption that there are only two agents in the economy, which leads him to rule out borrowing in the main section of the paper when he considers short-selling, and to rule out borrowing (and short-selling) at least for some agents in his Appendix.

In contrast, the very tractable setup of this paper allows a very natural treatment of short-selling, lending, securities lending, and borrowing at the same time. A number of new results come out of this analysis. Asset prices now reflect the opinion of pessimists, rather than that of optimists, for low disagreement. Endogenous rebate rates arise, which do not represent short-selling constraints, but instead equilibrium prices coming from the assignment mechanism. They are of comparable magnitude as that reported in the finance literature, that is a few tens of basis points (D'Avolio (2002)). All assets are potentially available for loans, but only a fraction of them are in equilibrium: short interest is endogenous: with reasonable values for disagreement, short interest is a few percentage points, as in the data.

Finally, short-selling reinforces most of the conclusions from the preceding analysis. Interest rates on bonds are higher than the returns to cash, not only because of the assignment mechanism, but also to incentivize natural short-sellers of the asset to lend rather than short.[12] Power law distributions for the leverage ratios of borrowers arise in Economy $\mathcal{E}_{\mathcal{S}}$ in the same way as in Economy $\mathcal{E}_{\mathcal{B}}$. Again, this contrasts with existing common wisdom, according to which conclusions from disagreement models are for the most part invalidated when short-selling is introduced.

4.1 Setup and Equilibrium

Economy $\mathcal{E}_{\mathcal{S}}$ has Borrowing Contracts like Economy $\mathcal{E}_{\mathcal{B}}$ (see Definition 1). On top of that, agents can also agree to collateralized Short-Sales Contracts with each other. A Short-Sales Contract is formally defined as follows.

Definition 4. (Short-Sales Contract, Economy $\mathcal{E}_{\mathcal{S}}$) A *Short-Sales Contract* $(\gamma)^s$ in economy $\mathcal{E}_{\mathcal{S}}$ is a promise of 1 unit of asset in period 1, collateralized by γ units of Cash, the *cash-collateral*.

Note that the normalization is this time done with respect to the loan amount, normalized to one unit of asset, not the collateral amount. This is because the loan is in terms of units of assets, and beliefs are expressed in these terms. But again, the normalization is without loss of generality. Abstracting from default risk, the Short-Sales Contract replicates the payoff of a Real Asset. Short-Sales Contracts are traded in an anonymous market at competitive price $q_s(\gamma)$, and payment is again only enforced by collateral, this time in the form of cash. The payoff of contract (γ) is given by:

$$\min\{\gamma, p_1\},$$

for a contract with cash-collateral amount γ.

Endogenous rebate rates. Short-Sales Contracts replicate the payoff of a real asset, if the collateral amount is sufficient. (in equilibrium, it will be according to the

[12] "Incentivizes" is an abuse of language here, used to help build an intuition. But markets are still anonymous, and prices are market clearing in a general equilibrium sense.

beliefs of agents trading them) The price at which they are sold by short-sellers is $q_s(\gamma)$, which in equilibrium will be lower than p: this wedge represents the cost of short-selling. This cost of shorting is in market practice expressed in terms of a rebate rate on the cash collateral kept with the lender of the security, whose return is lower than the return to cash. The rebate rate is here given by:

$$r_s(\gamma) = 1 - \frac{p - q_s(\gamma)}{\gamma}.$$

Equilibrium. In Economy $\mathcal{E}_{\mathcal{S}}$, agents choose their positions in Assets n^i_A, Borrowing and Short-Sales Contracts $n^i_B(.), n^i_S(.)$, and Cash n^i_C, so as to maximize their expected wealth in period 1 (W) according to their subjective beliefs p^i_1 about the Real Asset, subject to their budget contraint (BC), their collateral constraint (CC), and their cash-collateral constraint for Short-Sales Contracts (CCC):

$$\max_{(n^i_A, n^i_B(.), n^i_C), n^i_S(.))} n^i_A p^i_1 + \int_\phi n^i_B(\phi)\min\{\phi, p^i_1\}d\phi + n^i_C + \int_\gamma n^i_S(\gamma)\min\{\gamma, p^i_1\}d\gamma \tag{W}$$

$$\text{s.t.} \quad n^i_A p + \int_\phi n^i_B(\phi)q(\phi)d\phi + n^i_C + \int_\gamma n^i_S(\gamma)q_s(\gamma)d\gamma \leq 1 \tag{BC}$$

$$\text{s.t.} \quad \int_\phi \max\{-n^i_B(\phi), 0\}d\phi \leq n^i_A \tag{CC}$$

$$\text{s.t.} \quad \int_\gamma \max\{-\gamma n^i_S(\gamma), 0\}d\gamma \leq n^i_C \tag{CCC}$$

$$\text{s.t.} \quad n^i_A \geq 0, \quad n^i_C \geq 0$$

As in Economy $\mathcal{E}_{\mathcal{B}}$, a *Competitive Equilibrium* for Economy $\mathcal{E}_{\mathcal{S}}$ is a price p for the Real Asset and a distribution of prices $q(.)$ for all traded Borrowing Contracts (ϕ) and a distribution of prices $q_s(.)$ for all traded Short-Sales Contracts $(\gamma)^s$, and portfolios $(n^i_A, n^i_B(\phi), n^i_C, n^i_S(\gamma))$ for all agents i in the Real Asset, Borrowing Contracts, Cash, and Short-Sales Contracts such that all agents i maximize expected wealth according to their subjective beliefs (W), subject to their budget constraint (BC), the collateral constraint (CC), the cash-collateral constraint (CCC) and markets for the Real Asset, Borrowing Contracts, and Short-Sales Contracts clear:

$$\int_i n^i_A di = S, \tag{MC_A}$$

$$\forall\phi, \quad \int_i n^i_B(\phi)di = 0. \tag{MC_B}$$

$$\text{and} \quad \forall\gamma, \quad \int_i n^i_S(\gamma)di = 0. \tag{MC_S}$$

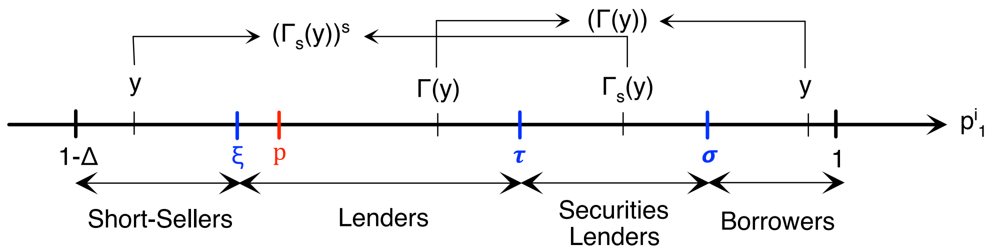
The next proposition shows that Economy $\mathcal{E}_{\mathcal{S}}$ can be characterized in a very similar way as the two preceding economies.

Proposition 4. *(Equilibrium of Economy $\mathcal{E}_{\mathcal{S}}$) A competitive equilibrium of Economy $\mathcal{E}_{\mathcal{S}}$ is described by a price p for the Asset, three thresholds $\xi \leq \tau \leq \sigma$, two strictly increasing matching functions $\Gamma(.)$ mapping $[\sigma, 1]$ onto $[\xi, \tau]$, and $\Gamma_s(.)$ mapping $[1-\Delta, \xi]$ onto $[\tau, \sigma]$, and two loan and short-sales pricing functions $Q(.)$ and $Q_s(.)$ such that:*

- *The space of agents' beliefs $[1-\Delta, 1]$ is partitioned through ξ, τ, and σ into four intervals:*
  - *Agents with beliefs $p_1^i \in [1-\Delta, \xi]$ (short-sellers) sell Short-Sales Contracts and invest in Cash.*
  - *Agents with beliefs $p_1^i \in [\xi, \tau]$ (lenders) buy Borrowing Contracts.*
  - *Agents with beliefs $p_1^i \in [\tau, \sigma]$ (securities lenders) buy Short-Sales Contracts.*
  - *Agents with beliefs $p_1^i \in [\sigma, 1]$ (borrowers) buy the Asset and sell Borrowing Contracts.*
- *Lenders with beliefs x buy Borrowing Contracts with face value x. Borrowers with beliefs y sell Borrowing Contracts with face value $\Gamma(y)$, at price $Q(y)$. Securities lenders with beliefs z buy Short-Sales Contracts with cash-collateral z. Short-sellers with beliefs y sell Short-Sales Contracts with cash collateral $\Gamma_s(y)$, at price $Q_s(y)$.*
- *p, ξ, τ, σ, $\Gamma(.)$, $\Gamma_s(.)$, $Q(.)$ and $Q_s(.)$ are such that:*

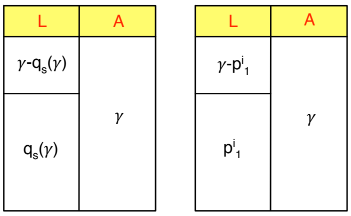
$$\begin{aligned} &(5a)\quad (p-Q)f(\Gamma)\,\Gamma' = Qf, &&(5b)\quad (\Gamma_s - Q_s)f(\Gamma_s)\,\Gamma_s' = Q_s f \\ &(5c)\quad (y-\Gamma)Q' = (p-Q)\Gamma', &&(5d)\quad (\Gamma_s - y)Q_s' = (Q_s - y)\Gamma_s'. \end{aligned} \tag{5}$$

$$\begin{aligned} &(6a)\quad 1 - F(\sigma) + F(\tau) - F(\xi) = p, \\ &(6b)\quad \Gamma(\sigma) = \xi, \quad (6c)\quad \Gamma(1) = \tau, \quad (6d)\quad \Gamma_s(1-\Delta) = \tau \quad (6e)\quad \Gamma_s(\xi) = \sigma \\ &(6f)\quad \frac{\xi}{Q(\sigma)} = \frac{\sigma-\xi}{\sigma - Q_s(\xi)}, \quad (6g)\quad \frac{\tau}{Q_s(1-\Delta)} = \frac{\tau}{Q(1)} \quad (6h)\quad \frac{\sigma}{Q_s(\xi)} = \frac{\sigma-\xi}{p-Q(\sigma)}. \end{aligned} \tag{6}$$



Proof. See Appendix C. □

Although the proof of this proposition is in Appendix C, it is useful to look at the balance sheet of short-sellers to understand the intuition for these Short-Sales Contracts. Below these balance sheets are represented in period 0 and in period 1.



$$\text{Return of Asset Lender} = \frac{p_1^i}{q_s(p_1^i)} = \frac{p_1^i}{p}\frac{p}{q_s(p_1^i)}.$$

$$\text{Return of Short-Seller} = \frac{\gamma - p_1^i}{\gamma - q_s(\gamma)}.$$

Short-sellers sell Short-Sales Contracts $(\gamma)^s$ at price $q_s(\gamma)$, by which they promise to give back one unit of the real asset in period 1 to the buyer of the Short-Sales Contracts (the securities lender). Because the short-seller must provide γ in total in terms of collateral, and can only raise $q_s(\gamma)$ through the sale of the security (less than p, because of the anticipation of a cost of short-selling, the rebate rate), he must finance $\gamma - q_s(\gamma)$ from his own funds. On the other hand, the return of the securities lender is composed of the normal return from the ownership of the real asset, plus an amount corresponding to the lower rebate rate on the collateral as the benchmark rate, which is a return $p/q_s(p_1^i)$.

4.2 Results

Proposition 4 introduces several new insights about short-selling in disagreement economies. There can be a coexistence of short-selling and lending, the price reflects the opinion of pessimists, interest rates even on the safest (lowest haircuts) bonds are strictly higher than the benchmark rate, and short interest and rebate rates are determined in equilibrium, even though no agent is constrained from short-selling, and short-selling entails no physical cost.[13]

Coexistence of Short-selling and Lending. A key insight from Proposition 4 is that lending and short-selling can very well coexist in equilibrium. A shortcoming of existing models with endogenous leverage, from Geanakoplos (1997) to Simsek (2013) is indeed that in these models, lenders would a priori rather sell short than lend if they were given this possibility.

Figure 17 shows for the case of a uniform distribution of beliefs how the population of investors splits into borrowers, securities lenders, cash lenders, and short-sellers, as a function of disagreement Δ.

[13]The conclusions from economies with impossible short-sales remain: hedonic interest rates, customization and fragmentation, Pareto distributions for leverage ratios, etc. For the sake of brievity, I do not come back to them.

[INSERT FIGURE 17 ABOUT HERE]

Price is Pessimists' valuation. Figure 17 also shows that in a Borrowing with Short-Sales Economy, the asset price is given by pessimists' valuation, for low disagreement. This result is reminiscent of that in De Long et al. (1990), where asset prices are lower than fundamentals because noise traders create an additional risk that other agents need to bear. De Long et al. (1990) have related their findings to the closed end fund discount, the equity premium, etc.

Equilibrium short interest. The fact that only a limited number of shares are sold short at any time has often been cited as evidence of short-sales constraints, or of the fact that there was no disagreement on financial markets. In this model, short-interest is of very small magnitude, even though no agent is constrained from selling-short, as long as he satisfies the collateral constraints. The short-interest is the percentage of Real Assets which is in period 0 on loan. It is therefore given by the following expression:

$$\text{Short Interest} = \frac{F(\sigma) - F(\tau)}{p}.$$

Figure 18 shows that the short interest is of comparable magnitude as the amount of disagreement, when the belief density function is uniform. Short interest can therefore be used as another tool to recover the amount of disagreement in an economy.

[INSERT FIGURE 18 ABOUT HERE]

A key intuition from this extension is thus that limited short-selling on financial markets may not be evidence that short-selling is limited by constraints (or that differences of opinion are absent), as was previously thought.

Interest rates on the safest securities. The conclusions from Section 2.3 concerning interest rates on bonds are in fact reinforced in the presence of short-sales. The reason is that lenders now earn larger than benchmark returns, not just because of the assignment process, but because lenders are natural short-sellers of the asset, and need to be compensated for not short-selling in equilibrium.

Figure 19 displays the equilibrium interest rates on bonds in Economy $\mathcal{E}_{\mathcal{S}}$. They comprise two components: the hedonic component, as in Economy $\mathcal{E}_{\mathcal{B}}$, together with a component which encourages lenders to lend rather than to short. These rates are again quantitatively very substantial, of the order of 200 bps when disagreement is 2%.

[INSERT FIGURE 19 ABOUT HERE]

These insights could potentially explain the findings in Gorton and Metrick (2012), showing that interest rates on several collateralized loans started to increase in 2007, just when shorting the US housing market was made possible.

Further from collateralized lending, the intuition that lenders need to be compensated for not short-selling is likely to be more general. It could help explain why rates on even investment grade AAA securities, whose default losses are no higher than 2bps, are often 50bps higher than that on similar maturity Treasury securities, offering the same liquidity service. High rates incentivize pessimists to buy bonds rather than short-sell the stock of the corresponding corporation.

Equilibrium rebate rates. D'Avolio (2002) studies the market for borrowing securities in detail and finds evidence for non zero rebate rates. This study has however been interpreted as evidence that short-sales constraints were somewhat present, but not commensurate with the ex-post potential gains from selling short in periods of irrational exuberance. This model shows that non-zero rebate rates are perfectly compatible with a model of unconstrained short-selling. Rebate rates might not be transaction costs, as previously thought, but may well be market clearing prices.

In this model, the source of rebate rates is twofold. First, lenders to short-sellers, who are relatively optimistic, must be willing to give up a levered bet on the risky asset, and rebate rates convince them to do so. Once the asset has been lent, it can no longer be used as collateral to lever up. Second, rebate rates are the symmetric of hedonic interest rates in the Borrowing Economy: they allow positive sorting between the most pessimistic short-sellers and the most pessimistic asset lenders. These more pessimistic asset lenders in equilibrium demand less cash-collateral, and thus allow short-sellers to lever up more.

Figure 20 shows that the order of magnitude of rebate rates is some tens of basis points, again in line with available empirical evidence.

[INSERT FIGURE 20 ABOUT HERE]

5 Empirical Applications

After presenting the baseline model, and two extensions, I now illustrate three potential applications of the model. The first one concerns homeowners' initial leverage ratios, as measured in the microdata from DataQuick. It shows that final borrowers' leverage ratio distribution contains useful information on the buildup of risk. A second takes a look at entrepreneurs' leverage, using the Survey of Consumer Finances. Finally, I apply the model to hedge funds' leverage.

5.1 Homeowners' Leverage (DataQuick microdata)

The first application is homeowners' leverage. This evidence suggests that monitoring ultimate borrowers' leverage ratio distribution could provide some information on the buildup of risk.

Data construction. DataQuick collects and digitizes public records from county register of deeds and assessor offices and provides a detailed transaction history of each property sold in the United States from 1988 to 2013. The data includes the price of the home exchanged in the transaction, as well as the first, second, and third mortgage loan amounts. From this dataset, I sum the values of the first, second, and third mortgages, and call the resulting new variable *Loan Amount*. If *Price* denotes the price of the transaction, the initial leverage ratio as defined in the model above is then given by:

$$\text{Initial Leverage Ratio} = \frac{\text{Price}}{\text{Price - Loan Amount}}.$$

Results. In Figure 21, I plot the cross section of leverage ratios of homeowners on a log survivor-log leverage ratio scale, corresponding to different months: October 1989, October 2001, October 2006 (before the crisis) and October 2012 (after the crisis). The dataset has 200000 to 500000 observations of new loans each month.

[INSERT FIGURE 21 ABOUT HERE]

A first observation is that the resulting distribution of leverage ratios takes the form of an approximate power law in the upper tail.[14] A second is the evolution of this leverage ratio distribution, and in particular the decrease in the Pareto coefficient during the years leading up the the financial crisis. The evolution of these Pareto coefficients can be eyeballed on Figure 21, which shows a decrease in the Pareto coefficient in October 2001, October 2006, and then a rise in this Pareto coefficient in October 2012. More data is available in an online video (which shows the universe of the Dataquick data at my disposal), and the evolution of this distribution before the financial crisis is quite striking. The solid line on Figure 22 shows the evolution of this Pareto coefficient, when measured over the top 25th percentile of borrowers.

[INSERT FIGURE 22 ABOUT HERE]

According to the model presented before, and as suggested on Figure 14 in particular (which should be compared to the change between October 2001 and October 2006 on Figure 21), such an evolution can arise when pyramiding lending arrangements, or tranching, enable optimists to express themselves more, and to borrow from more and more investors. Again, final borrowers' leverage ratios then result both from their own borrowing from banks, but also on how much banks themselves borrow against these loans - because of these two sources of fat tailedness, the Pareto coefficient intuitively

[14] However, one can note a deviation from this Pareto distribution which likely results from regulation, and that leads homeowners to bunch around some regulatory threshold - for example, to make this loan a conforming loan.

decreases. As Figure 13 compared to Figure 3 has shown, a decreasing Pareto coefficient can thus be seen as a signature of the asset price representing then more and more the opinion of a very optimistic investor. This price has then higher odds to be “too high”. To that extent a decreasing Pareto coefficient is a sign of growing risk. Figure 22, and the negative correlation between the Pareto coefficient and house prices, seem to confirm this hypothesis.

In a nutshell, Figure 21 and Figure 22 thus suggest that monitoring the leverage ratio distribution could have alerted policymakers that prices were diverging a lot from fundamentals in the runup to the financial crisis. With the benefit of hindsight, this decrease in the Pareto coefficient likely was the signature of a shadow banking sector which was allowing fastly growing tranching and pyramiding, and which took economists by surprise in 2008.

One could also very well use this method to estimate the distribution of beliefs at a more local level, and see whether it correlates with land availability, past movement of housing prices, or other potential predictors of housing prices. This would allow to put different theories of asset price bubbles to the test. Such an empirical investigation is left for future research.

5.2 Entrepreneurs' Leverage in the Survey of Consumer Finances (SCF)

A second potential application of the model concerns entrepreneurial leverage.

The leverage ratio distribution of Entrepreneurs in the SCF. Empirically, the SCF allows to compute a measure for the leverage of entrepreneurs. More precisely, I compute the following ratio:

$$\text{Leverage Ratio} = \frac{\text{Price}}{\text{Price} - \text{Outstanding Loan Amount}},$$

where “Price” corresponds to variable X3130, which is the answer to the following question: “If you sold the business now, what would be the cost basis for tax purposes (of your share of this business)?”. The cost basis for tax purposes is the amount of the original investment (or the value when it was received) plus additional investments minus depreciation. “Outstanding Loan Amount” corresponds to variable X3121. Figure 23 shows the log survivor function as a function of log leverage, and a distribution that is quite close to Pareto. The underlying level of disagreement, assuming this distribution was generated by the model above in the case of a Borrowing Economy, is $\Delta \approx 20\%$.

One caveat however is that unlike what I obtained for housing in DataQuick micro-data, the measure here is more indirect, as I do not measure initial leverage. One can however restrict the set of entrepreneurs who have started their business less than three years ago, and obtain a roughly similar picture. Another is that one could worry that assets in which entrepreneurs invest in are too heterogenous, so that the model developed above does not apply. As long as the number of borrowers is large compared to the number of different investment possibilities, individual Pareto distributions would however aggregate into an overall power law distributions.

[INSERT FIGURE 23 ABOUT HERE]

Low returns to entrepreneurship. The model provides a new explanation for why the returns to entrepreneurship are low on average (see Moskowitz and Vissing-Jørgensen (2002)). In this model, entrepreneurs / borrowers are too optimistic on average endogenously.[15] Models of borrowing constraints resulting from information asymmetries would predict the opposite.

5.3 Hedge Funds' Leverage (TASS Hedge Fund Database)

A final illustration of the model is given using the TASS Hedge Fund Database, which contains 50% of the universe of hedge funds. Hedge Funds report their average leverage monthly to this database, on a voluntary basis. The distribution of average leverage ratios that these hedge funds report is shown on Figure 24. The point estimate for the regression of the log survivor function on log leverage is -1.95, with an R^2 close to 98%, and a standard error of 0.2 (computed using Gabaix and Ibragimov (2011)'s method).

Again, one could worry that the assets hedge funds invest in are much less homogenous than housing. One can however condition on the strategies that the hedge funds are using. One disadvantage is that some strategies do not comprise so many hedge funds in the TASS database so as to be able to compute such a distribution. However, quite a few hedge funds are classified under the label "Fund of Funds", and "Long/Short Equity Hedge". For these, the Pareto distribution is also a good approximation and the coefficients are -2.16 and -3.08 respectively. As for entrepreneurship, as long as the number of hedge funds using each strategy is large compared to the number of different strategies ("the asset"), individual Pareto distributions would however aggregate into an overall power law distributions. Whether this is a good assumption is an empirical issue.

[INSERT FIGURE 24 ABOUT HERE]

Disagreement Δ can then be calibrated to the data, for example through the maximum or the minimum leverage in the upper tail and in August 2006, is estimated to be around $\Delta \approx 1.9\%$ through both methods.

[15] The no short-sales assumption is a natural one in this case, as there are no markets to bet on the failure of a specific entrepreneur.

A key insight from the model is that these high leverage ratios are justified by the fact that hedge funds trade on assets on which agents disagree very little (for example, through arbitrage on very similar securities). The model allows to understand why their realized returns can become very heterogenous, skewed and fat tailed, depending on whether they were right or not.

6 Concluding Remarks

This paper has developed a stylized model of borrowing and lending with endogenous leverage. Some new insights are qualitative, and are therefore likely to survive in richer models. One of them is that an economy with disagreement generates very large amounts of equilibrium leverage, even from vanishingly small heterogeneity in beliefs. Instead of a no-trade outcome, there is a natural tendency for financial markets to be characterized by some agents taking very large and heterogenous positions.

Hedonic interest rates are another of these qualitative insights. They arise in the present model from the competition of borrowers for certain types of lenders, in order to achieve the highest possible leverage, and are therefore likely to be present in more complex models. The insight that hedonic interest rates can be disconnected from expected default risk can be potentially important given the importance of fixed income securities in modern financial markets. On the normative front, it shows that financial institutions buying high yield fixed income securities are not necessarily engaging in risk shifting, and thus that more "skin in the game", such as higher capital requirements for banks, might not make them less prone to crises. Hedonic interest rates also explain why pyramiding lending arrangements exist, and why they are not redundant. The modeling of these chains of lending is new and potentially important given their centrality in the recent financial crisis.

Another qualitative insight from the model, which is likely to carry through in more general environments, is that speculation requires very specific contracts, tailored to the particular beliefs of every financial market participant. While the search literature following Duffie et al. (2005) has assumed an exogenous OTC structure for derivatives or repo markets, the model can rationalize this microstructure as resulting from a sorting mechanism in a purely competitive market. Understanding the rationale for OTC markets is important because they apply to a very large set of financial assets and they received significant attention during the recent financial crisis. For example, the model could provide a framework to assess the ongoing regulatory efforts trying to push banks' bilateral trades onto exchanges.

Other results in this paper are more quantitative. Among them features prominently the Pareto distribution for leverage ratios, which is shown to arise under very mild assumptions on the underlying distribution of beliefs. It is also shown to describe quite well some real-world markets, such as the initial leverage ratio on new loans of US homeowners, and that monitoring the distribution of homeowners' leverage ratio could have alerted regulators in real time of a growing shadow banking system. More work is however needed to know whether or not this Pareto distribution is robust to other assumptions on preferences, or whether it relies on risk neutrality. The answer is not obvious a priori, notably because borrowers are shown to also have very large expected returns from leveraging.

A dynamic version of this model also could have interesting implications for finance. When the wealth distribution becomes more skewed towards optimists, leverage rises as optimists with the same beliefs become more numerous; and conversely when the fat tail of optimistic investors is wiped out because of a crisis. In other words, the model could generate an increase in margins during crises without an assumption of "scary bad news", an important stylized fact that models with a single leverage ratio do not generate (Geanakoplos (2003)). Similarly, short-sellers could be wiped out during an asset price bubble, if they turn out to be "right too soon". (see Lamont and Stein (2004)) Another example would be to investigate the determinants of Leveraged Buy-Outs (LBOs), where leverage does play a big role and easing of credit market conditions seems to be associated to overpricing (see for example Axelson et al. (2013)), something that the present model would predict.

Finally, a model of this type could be useful to study the heterogenous returns to entrepreneurship as well as top income inequality coming from the banking industry more generally. This paper gives a new intuition for why returns to entrepreneurship or to specific trading strategies are skewed to the right in a Paretian manner, regardless of the underlying level of disagreement. These important questions are left for future research.

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A Completing the Proof of Proposition 1

A.1 Prices of non-traded Borrowing Contracts

Proposition 1 and its proof in the main text have focused on the price of Borrowing Contracts which are traded in equilibrium by borrowers and lenders. However, one must check that prices of other Borrowing Contracts exist, which discourage borrowers from supplying and lenders from demanding such Borrowing Contracts. As is usual in this literature (see Simsek (2013)), one should not expect the price of non-traded Borrowing Contracts to be unique. A sufficient condition for these prices to exist that the following two conditions are satisfied:

$$r'(\xi) > 1 \quad \text{and} \quad r'(\tau) < \frac{1}{p-\xi}.$$

It is indeed the case in equilibrium. They result from rewriting the assignment equation (1b) in terms of the return function using $q(\phi) = \phi/r(\phi)$:

$$\frac{y-\phi}{p-q(\phi)} = \frac{1}{q'(\phi)} \quad \Rightarrow \quad \frac{r'(\phi)\phi}{r(\phi)} = \frac{y-pr(\phi)}{y-\phi}.$$

For $\phi = \xi$ we thus we have that:

$$r'(\xi) = \frac{r(\xi)}{\xi}\frac{\tau - pr(\xi)}{\tau - \xi} = \frac{1}{\xi}\frac{\tau - p}{\tau - \xi} > 1 \quad \text{since} \quad 1 > p > \xi.$$

For $\phi = \tau$ this gives:

$$r'(\tau) = \frac{r(\tau)}{\tau}\frac{1-pr(\tau)}{1-\tau} < \frac{r(\tau)}{\tau} = \frac{\tau-\xi}{\tau}\frac{1}{p-\xi} < \frac{1}{p-\xi} \quad \text{since} \quad r(\tau) = \frac{\tau-\xi}{p-\xi} > \frac{\tau}{p}.$$

Why are these two inequalities sufficient? One can then construct the prices $q(\phi)$ for non traded Borrowing Contracts (ϕ) with $\phi < \xi$ and $\phi > \tau$, such that the return $r(\phi)$ is linear with a slope strictly comprised between 1 and $r'(\xi)$ for $\phi < \xi$, and linear with a slope strictly comprised between $r'(\tau)$ and $\frac{1}{p-\xi}$ for $\phi > \tau$. Contracts with $\phi < \xi$ are then not demanded by any lender or cash investors because they have negative return, and are not supplied by any borrower because their return does not decrease enough compared to the diminished leverage they offer. Contracts with $\phi > \tau$ are similarly not demanded by any lender because they default for sure for all existing lenders and are more expensive than lower leverage ratio loans. Contracts with $\phi > \tau$ are not supplied by any borrower either as the extra return they would then give is too high.

Moreover, the first inequality guarantees that cash investors with beliefs $p_1^i = \xi^-$ (infinitesimally close to the left of ξ) do not want to lend rather than invest in cash, and symmetrically that lenders with beliefs $p_1^i = \xi^+$ do not want to invest in cash rather than lend. The second inequality guarantees that lenders with beliefs $p_1^i = \tau^-$ do not want to be borrowers instead of lenders. Indeed, their return decreases less fast than the levered return $\frac{p_1^i - \xi}{p-\xi}$ they would get from leveraging with contract (ξ): from an envelope condition, one can neglect the corresponding change in the face value of the Borrowing Contract. Symmetrically, it also guarantees that borrowers with beliefs $p_1^i = \tau^+$ do not want to lend rather than take a levered bet.

A.2 Formal Expressions for Agents' Porfolios

Portfolios of agents in Proposition 1 are only defined implicitly. To match the level of formalism used in Definitions 1 and 2, one needs to use mathematical distributions because portfolio problem consists in choosing between a continuum of commodities. In the following, $\delta_x(.)$ denotes the Dirac measure with mass point at x. The following statements complete Proposition 1 in characterizing equilibrium in this economy.

- Portfolios of cash investors with beliefs such that $p_1^i \in [1-\Delta, \xi]$ are:

$$n_A^i = 0, \qquad n_B^i(.) = 0, \qquad n_C^i = 1.$$

- Portfolios of lenders with beliefs such that $p_1^i \in [\xi, \tau]$ are:

$$n_A^i = 0, \qquad n_B^i(.) = \frac{1}{q(p_1^i)}\delta_{p_1^i}(.), \qquad n_C^i = 0.$$

- Portfolios of borrowers with beliefs such that $p_1^i \in [\tau, 1]$ are:

$$n_A^i = \frac{1}{p - q(\phi')}, \qquad n_B^i(.) = -\frac{1}{p - q(\phi')}\delta_{\phi'}(.), \qquad n_C^i = 0,$$

$$\text{with } \phi' \text{ given by: } \quad \phi' = arg \max_{\phi} \frac{p_1^i - \phi}{p - q(\phi)}.$$

Note that equation (2a) obtains from aggregating all consumers' budget constraint (at equality), using market clearing for the real asset (MC_A) and all Borrowing Contracts (MC_B):

$$\int_i n_A^i p di + \int_i \int_\phi n_B^i(\phi) q(\phi) d\phi di + \int_i n_C^i di = \int_i di \quad \Rightarrow \quad p + \int_i n_C^i di = 1.$$

Because only agents i with $p_1^i \in [1-\Delta, \xi]$ buy cash in quantity $n_C^i = 1$,we get equation (2a) as:

$$\int_i n_C^i di = F(\xi) \quad \Rightarrow \quad p = 1 - F(\xi). \tag{2a}$$

Finally, note that the first differential equation (1a) obtains with these notations from (MC_B):

$$\begin{aligned}
\int_i n_B^i(\phi) di = 0 \quad &\Rightarrow \quad -\frac{1}{p - q(x)} f(y) dy + \frac{1}{q(x)} f(\Gamma(y)) d\Gamma(y) = 0 \\
&\Rightarrow \quad -\frac{1}{p - Q(y)} f(y) dy + \frac{1}{Q(y)} f(\Gamma(y)) d\Gamma(y) = 0 \\
\int_i n_B^i(\phi) di = 0 \quad &\Rightarrow \quad (p - Q) f(\Gamma)\Gamma' = Qf.
\end{aligned} \tag{1a}$$

One can wonder from this reasoning, as well as from the proof in the main part of the paper, why both the market clearing equation for the real asset, and the market clearing equations for financial contracts have been used, while one equation should be redundant by Walras' law. The answer is that the market clearing equations for financial contracts were written for intervals, so that one of them is missing.

B Completing the Proof of Lemma 5 - Closed Form Expressions for the cutoffs

A closed form expression for p, ξ and τ in the Borrowing Economy and when the density is uniform with heterogeneity parameter Δ obtains as follows. If f is uniform, then $F(.)$ is linear with slope $1/\Delta$ so that the market clearing equation (2a) writes $\xi = 1 - p\Delta$. Because $Q'(p-Q)$ is constant, from $\left((p-Q)^2\right)'' = 0$, we have:

$$\begin{aligned}
&Q'(\tau)\left(p - Q(\tau)\right) = Q'(1)\left(p - Q(1)\right) \quad \Rightarrow \quad \frac{Q(\tau)}{1 - \Gamma(\tau)}\left(p - Q(\tau)\right) = \frac{Q(1)}{1 - \Gamma(1)}\left(p - Q(1)\right) \\
&\Rightarrow \quad \frac{\xi}{\tau - \xi}(p - \xi) = \frac{\tau(p-\xi)}{(\tau - \xi)((1-\tau)}\left(p - \frac{\tau(p-\xi)}{\tau - \xi}\right) \quad \Rightarrow \quad \frac{\tau - \xi}{\tau - p} = \frac{\tau}{1 - \tau}.
\end{aligned}$$

Using $\xi = 1 - p\Delta$, one can express p as a function of τ:

$$(1-\tau)(\tau - 1 + p\Delta) = \tau(\tau - p) \quad \Rightarrow \quad p = \frac{2\tau^2 - 2\tau + 1}{\Delta + (1-\Delta)\tau}.$$

Using the integration of $\left((p-Q)^2\right)'' = 0$ in the main proof of Corollary 5 we also have:

$$(p-Q(\tau))^2 = (p-Q(1))^2 + 2Q(1)\,(p-Q(1)) \quad \Rightarrow \quad (p-Q(\tau))^2 = p^2 - Q(1)^2.$$

Then, we have:

$$p-\xi = p-(1-\Delta p) = \frac{(1+\Delta)\tau^2 + (1+\Delta)(1-\tau)^2 - \tau - (1-\tau)\Delta}{\Delta+(1-\Delta)\tau} = \frac{(2\tau-1)\,[(1+\Delta)\tau-1]}{\Delta+(1-\Delta)\tau}$$
$$\tau-\xi = \tau-1+\Delta p = \frac{(\tau-1)\Delta + (1-\Delta)\tau^2 - (1-\Delta)\tau + 2\left(2\tau^2-2\tau+1\right)}{\Delta+(1-\Delta)\tau} = \frac{(1+\Delta)\tau-1}{\Delta+(1-\Delta)\tau}\tau.$$

Therefore, using equation (2e):

$$Q(1) = \tau\frac{p-\xi}{\tau-\xi} = 2\tau-1.$$

Equation $(p-\xi)^2 = p^2 - Q(1)^2$ thus writes:

$$(2\tau-1)^2\,[(1+\Delta)\tau-1]^2 = (2\tau^2-2\tau+1)^2 - (2\tau-1)^2\,[\Delta+(1-\Delta\tau]^2$$
$$\Rightarrow \quad (2\tau-1)^2(1+\Delta^2)(2\tau^2-2\tau+1) = \left(2\tau^2-2\tau+1\right)^2.$$

Because $2\tau^2-2\tau+1 = \tau^2+(1-\tau)^2 \neq 0$ this implies:

$$(2\tau-1)^2(1+\Delta^2) = 2\tau^2-2\tau+1 \quad \Rightarrow \quad \tau^2-\tau+\frac{2\Delta^2}{4(1+2\Delta^2)} = 0$$
$$\Rightarrow \quad \tau = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{1+2\Delta^2}}\right).$$

If τ is the lowest solution then from the implied value of p one can show easily that $\tau - p < 0$. This is a contradiction because borrowers would then be expecting negative excess returns on the asset. So the highest solution is the equilibrium one. One then gets a closed form expression for p and ξ as a function of Δ as well, so that:

$$\tau = \frac{1}{2} + \frac{1}{2\sqrt{1+2\Delta^2}} \qquad p = \frac{2(1+\Delta^2)}{1+\Delta+2\Delta^2+2\Delta^3+(1-\Delta)\sqrt{1+2\Delta^2}} \qquad \xi = 1-\Delta p.$$

C Borrowing with Short-Sales Economy - Proof of Proposition 4

Because of the linearity of the problem, whenever a short-seller finds it optimal to sell short, he will use all his cash as collateral to short as much as he can. Therefore, when using Short-Sales Contract $(\gamma)^s$ with cash collateral γ, the short-seller gets $q_s(\gamma)$ out of selling the contract, and so needs to contribute $\gamma - q_s(\gamma)$ of his personal funds to the purchase. He will be able to purchase a number $1/(\gamma - q_s(\gamma))$ of these contracts. On the asset side of his balance sheet, he gets the return on cash, which is γ per unit, and on the liability side buys back one unit of the contract to give it to the securities lender. Thus the Short-Seller chooses $(\gamma)^s$ to maximize:

$$\max_{\gamma} \frac{\gamma - p_1^i}{\gamma - q_s(\gamma)}.$$

The optimality condition thus gives equation (5d):

$$1 - q_s'(\gamma) = \frac{\gamma - q_s(\gamma)}{\gamma - p_1^i}$$
$$\Rightarrow \quad \forall y \in [1-\Delta, \xi], \quad Q_s'(y)\,(\Gamma_s(y) - y) = \Gamma_s'(y)\,(Q_s(y) - y).$$

The market clearing equation for Short-Sales Contracts leads to equation (5b):

$$\begin{aligned}
&\frac{f(x)dx}{Q_s(y)} = \frac{f(y)dy}{x - Q_s(y)} \\
&\quad \Rightarrow \quad \forall y \in [1-\Delta, \xi], \quad (\Gamma_s(y) - Q_s(y))\, f\,(\Gamma_s(y))\, \Gamma_s'(y) = Q_s(y) f(y).
\end{aligned}$$

The expressions concerning Borrowing Contracts, and leading to equations (5a) and (5c) are similar as those in Proposition 1. The reason why the cash collateral value is equal to the beliefs of securities lenders also arises in an exact symmetric way as for lenders. Finally, we need to prove the eight algebraic equations given by equation (6) to complete the proof.

Market clearing for the Real Asset implies that lenders' and borrowers' fund end up invested in the real asset, which is equation (6a):

$$1 - F(\sigma) + F(\tau) - F(\xi) = p.$$

Positive sorting of borrowers and lenders on the one hand, and securities lenders and short-sellers on the other, gives equations (6b), (6c), (6d) and (6e):

$$\Gamma(\sigma) = \xi, \qquad \Gamma(1) = \tau, \qquad \Gamma_s(1-\Delta) = \tau, \qquad \Gamma_s(\xi) = \sigma.$$

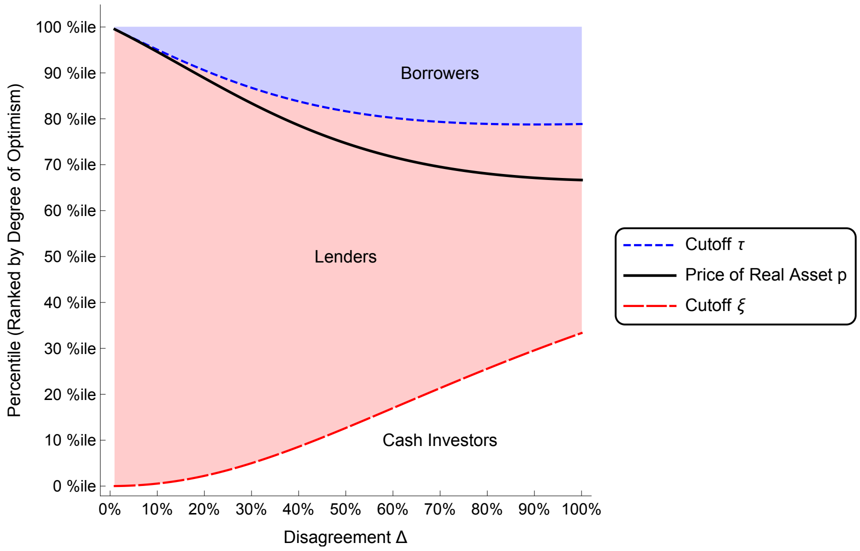
Finally, indifference for agents with beliefs ξ, τ and σ respectively imply equations (6f), (6g), and (6h):

$$\begin{aligned}
\frac{\xi}{q(\xi)} = \frac{\sigma - \xi}{\sigma - Q_s(\xi)} \quad &\Rightarrow \quad \frac{\xi}{Q(\sigma)} = \frac{\sigma - \xi}{\sigma - Q_s(\xi)} \\
\frac{\tau}{p}\frac{p}{q_s(\tau)} = \frac{\tau}{q(\tau)} \quad &\Rightarrow \quad \frac{\tau}{Q_s(1-\Delta)} = \frac{\tau}{Q(1)} \\
\frac{\sigma}{p}\frac{p}{q_s(\sigma)} = \frac{\sigma - \xi}{p - Q(\sigma)} \quad &\Rightarrow \quad \frac{\sigma}{Q_s(\xi)} = \frac{\sigma - \xi}{p - Q(\sigma)}.
\end{aligned}$$

D Figures

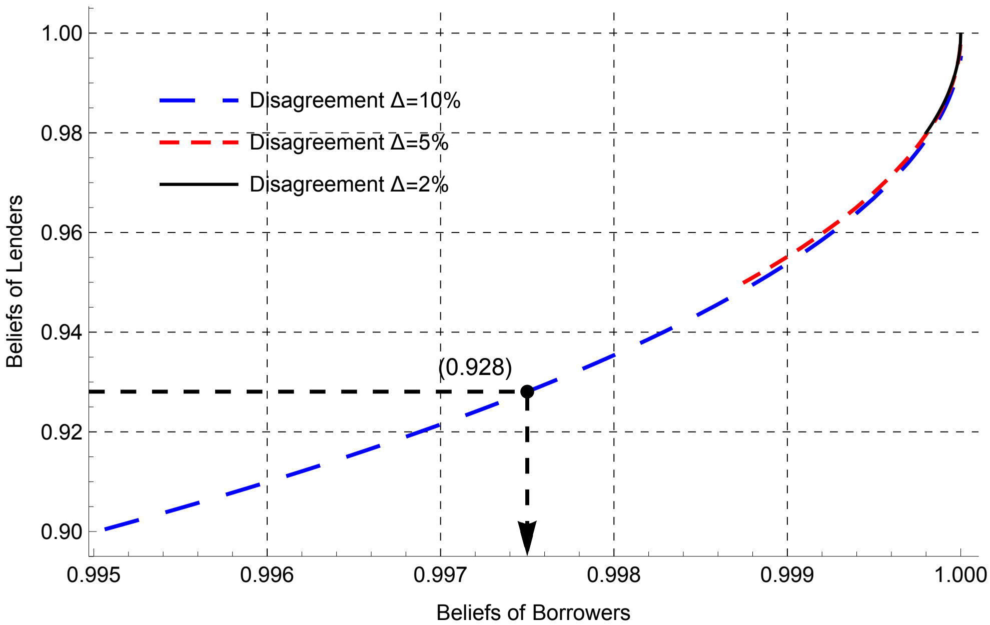
D.1 Borrowing Economy

Figure 3: **Investment Type**, f = Uniform



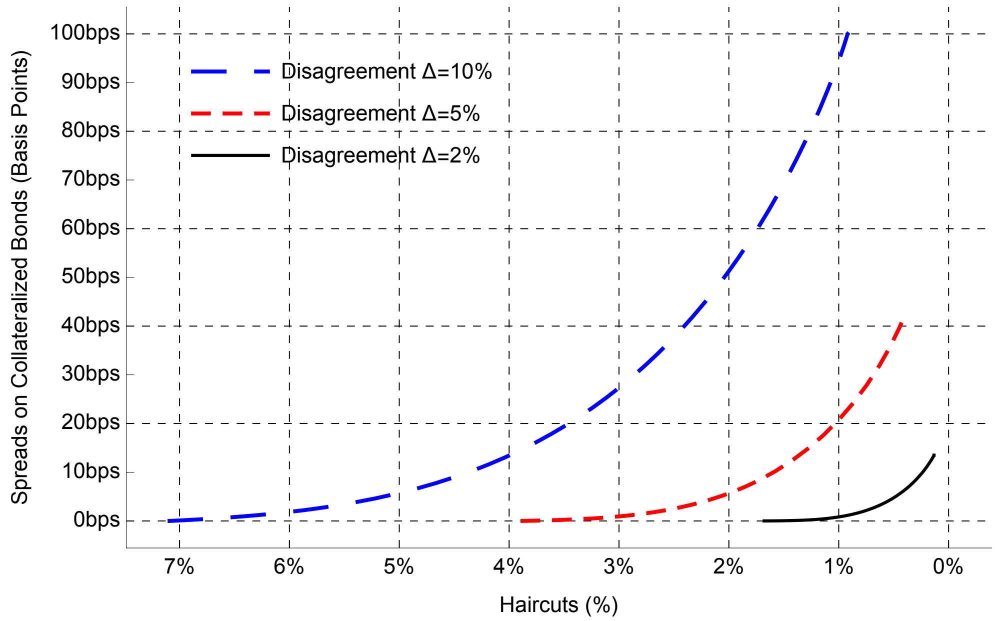
Note: The solid line represents the optimism of the agent who the price represents on average.

Figure 4: **Assignment** of Borrowers to Lenders, f = Uniform



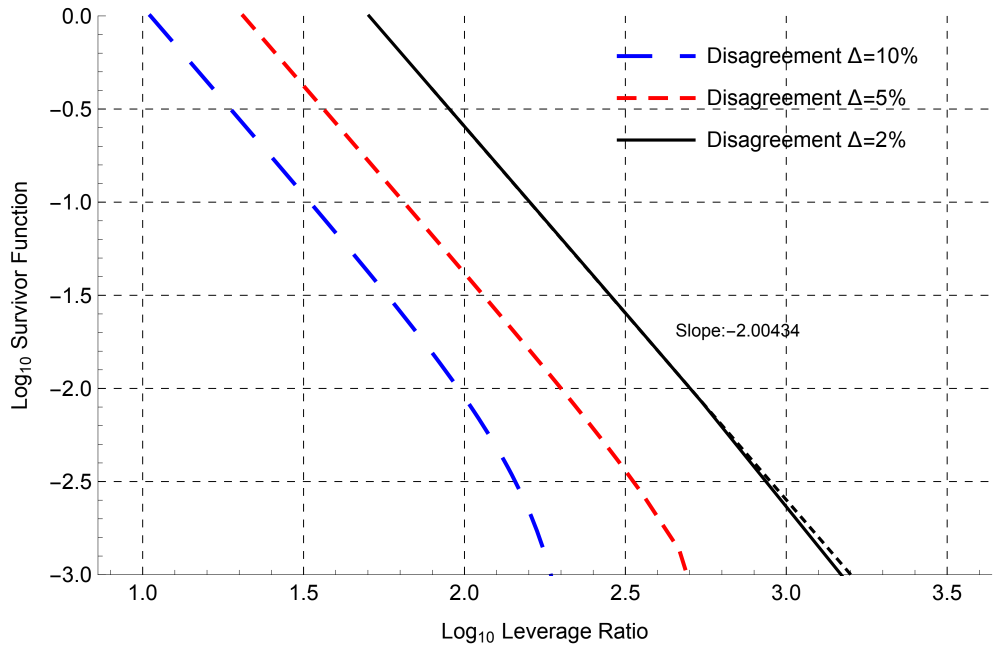
Note: In the case where $\Delta = 10\%$, lender with beliefs 0.928 chooses Borrowing contract (0.928) with face value 0.928, and a borrower with beliefs 0.9984 also chooses (0.928), given the price of all bonds. Thus lender with $p_1^i = 0.928$ is effectively lending to borrower with $p_1^i = 0.9984$ through their respective choices.

Figure 5: **Hedonic Spreads**, f = Beta(3, 3)



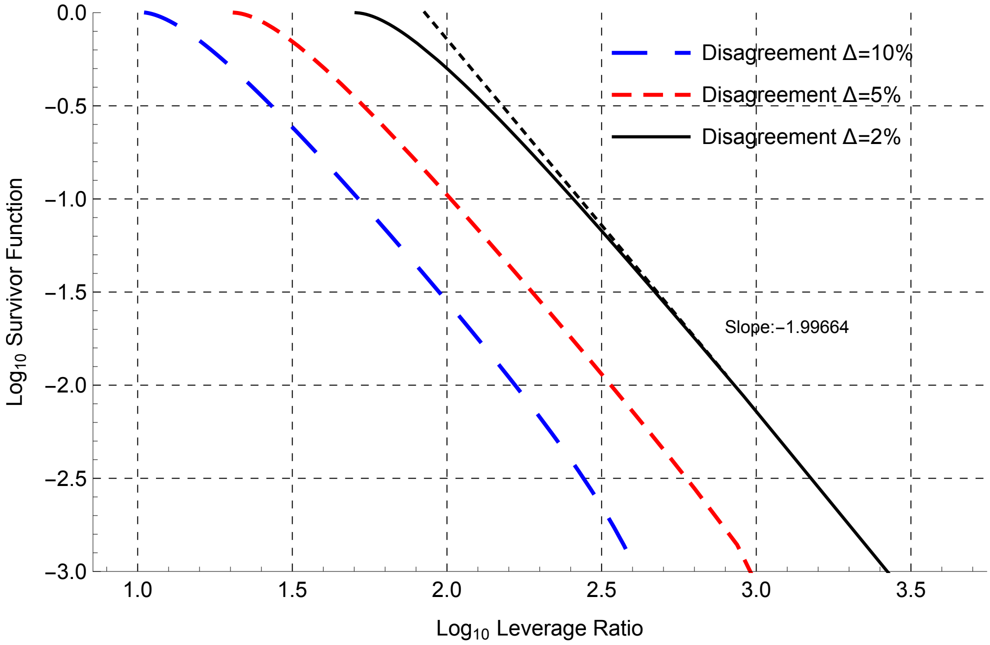
Note: Hedonic spreads can be substantial. With 10% disagreement and $f = \text{Beta}(3, 3)$, the standard deviation of beliefs is only 1.89%, while hedonic spreads can be as large as 100 bps=1%.

Figure 6: **Leverage Ratio Distribution** of Borrowers, Pareto Plot, f = Uniform



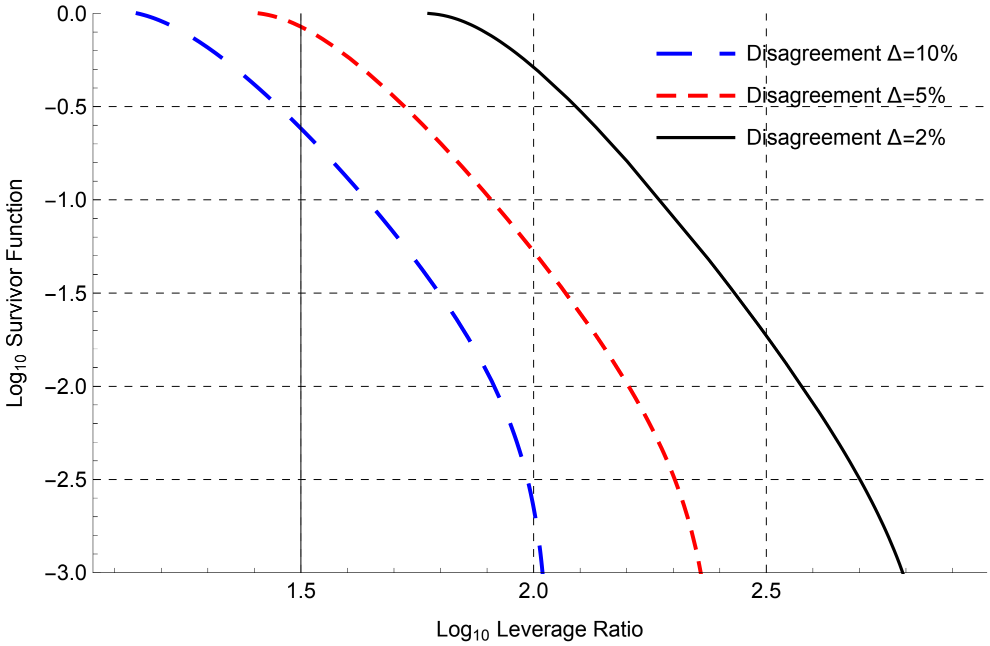
Note: The coefficients on these linear functions are equal to -2, so the tail index of the truncated Pareto is 2. The small deviation from Pareto in the very upper tail results from the truncation.

Figure 7: **Leverage Ratio Distribution** of Borrowers, Pareto Plot, f = Increasing



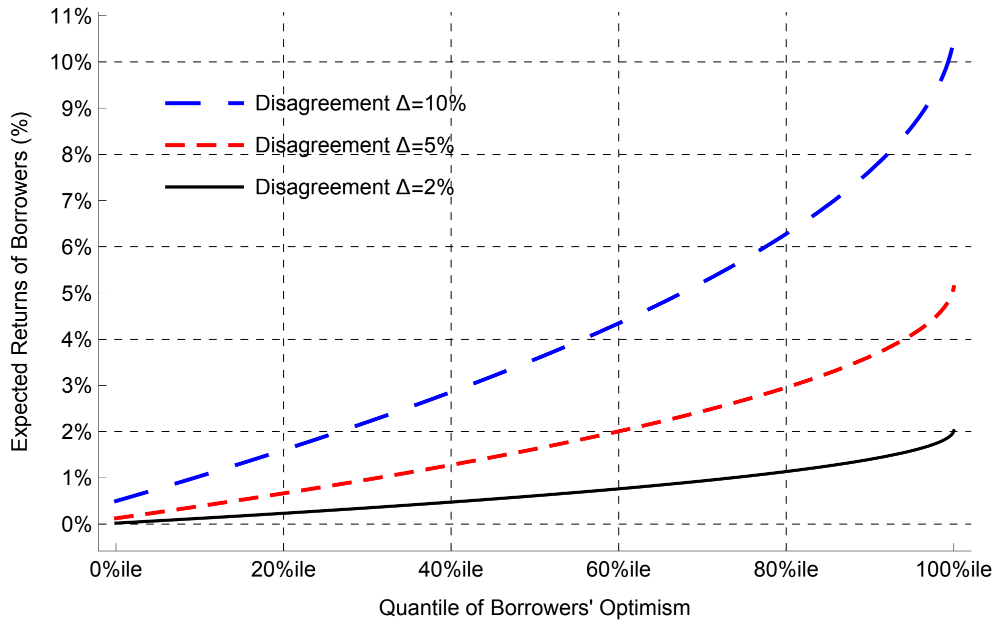
Note: This Figure illustrates that the distribution of leverage ratios in the upper tail does not depend on the underlying distribution of beliefs, as long as it is bounded away from zero. In the upper tail, the log survivor function if still linear as a function of the log leverage ratio, with a coefficient equal to -2.

Figure 8: **Leverage Ratio Distribution** of Borrowers, Pareto Plot, f = Beta$(3,3)$



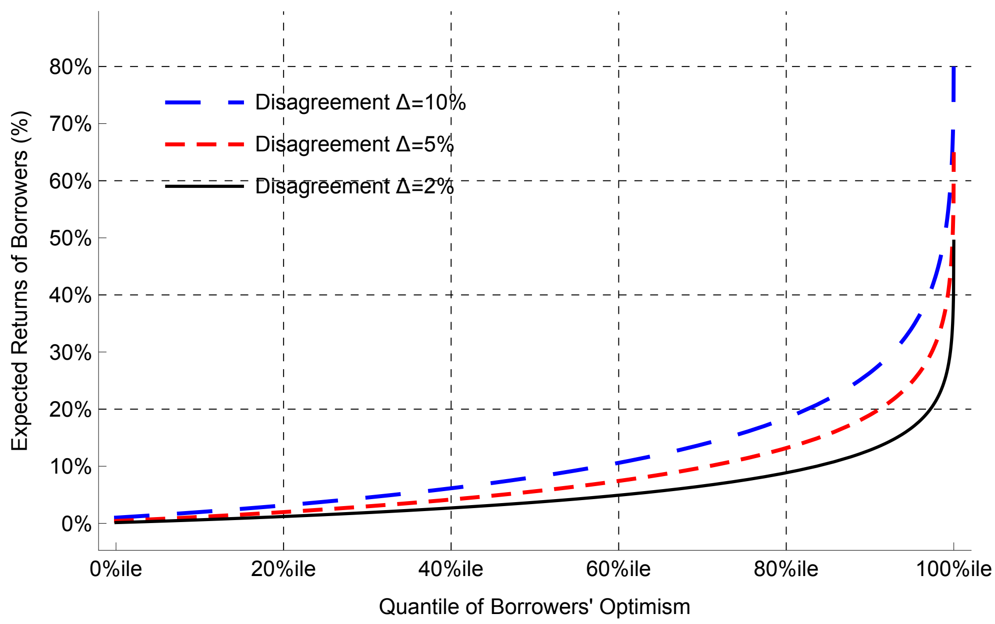
Note: This Figure illustrates that the distribution of leverage ratios in the upper tail is also Pareto distributed, albeit with a Pareto coefficient equal to $2+\rho$.

Figure 9: **Expected Excess Returns** of Borrowers, f = Uniform



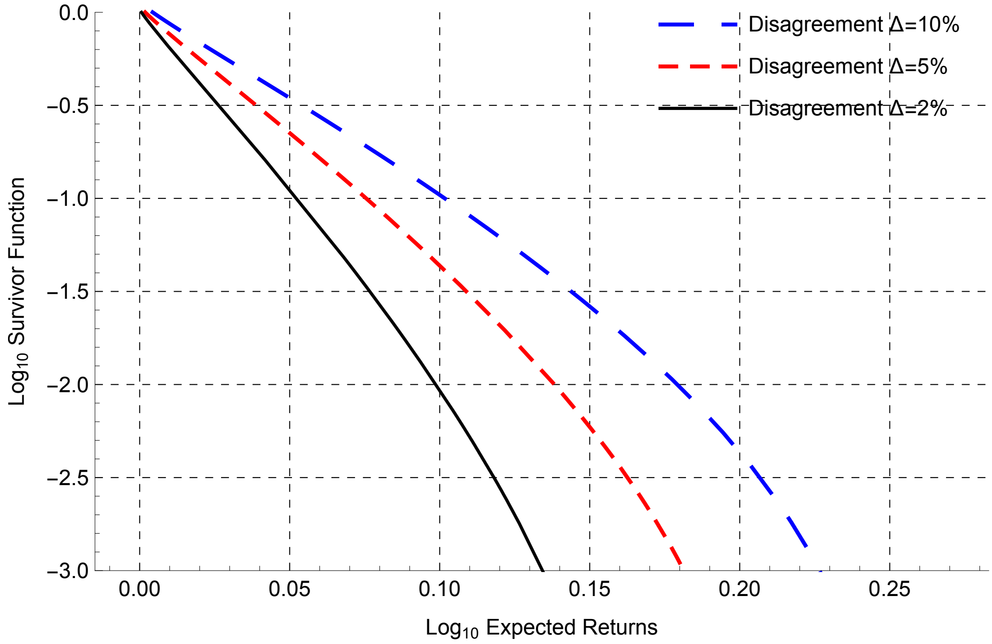
Note: Although leverage is very heterogenous across borrowers, expected returns are quite homogenous with $\rho = 0$, because of competition between borrowers for the most optimistic lenders, and for bidding up the asset.

Figure 10: **Expected Excess Returns** of Borrowers, f = Beta$(3, 3)$



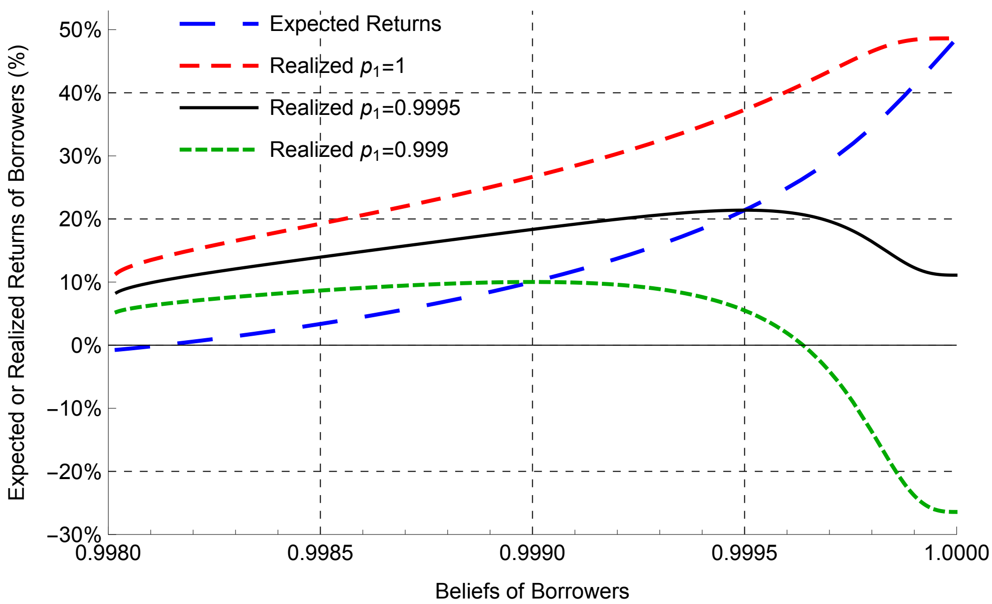
Note: When $\rho > 1$ (here $\rho = 2$), expected returns are much more heterogeneous across borrowers because competition between borrowers for lenders and the asset is less fierce.

Figure 11: **Expected Returns** of Borrowers, Pareto Plot, $f = \text{Beta}(3,3)$



Note: Again, when $\rho > 1$, expected returns display a Pareto like behavior. This comes from the envelope condition.

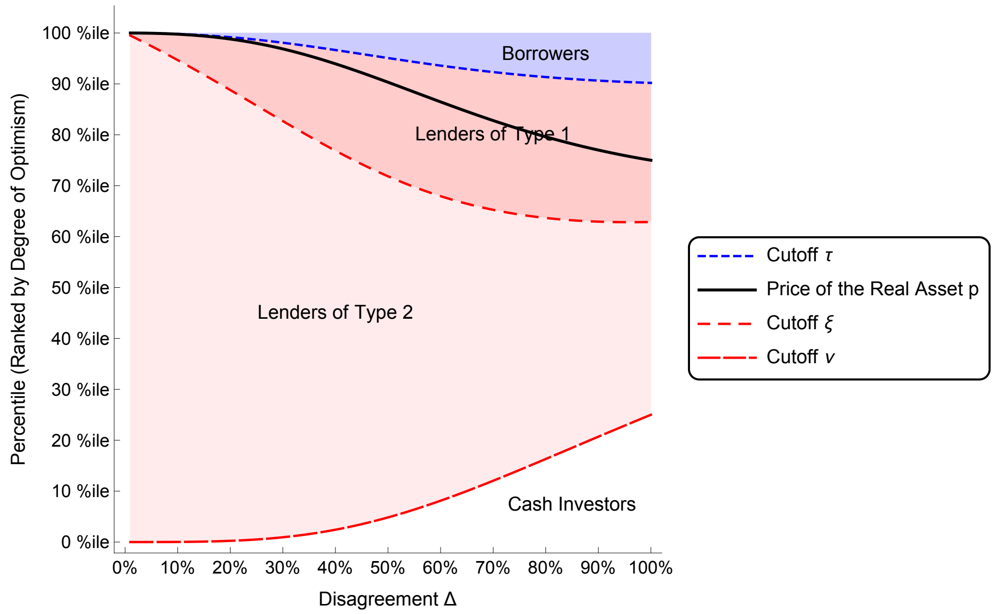
Figure 12: **Expected and Realized Returns of Borrowers**, $f = \text{Beta}(3,3)$, $\Delta = 2\%$



Note: When the realization of p_1 is interior, there is a non monotonic cross-sectional relationship between leverage and ex-post returns. Even when the return to the asset is better than the return to cash ($p_1 > p$), more leverage does not necessarily give a higher return, as borrowing costs are higher.

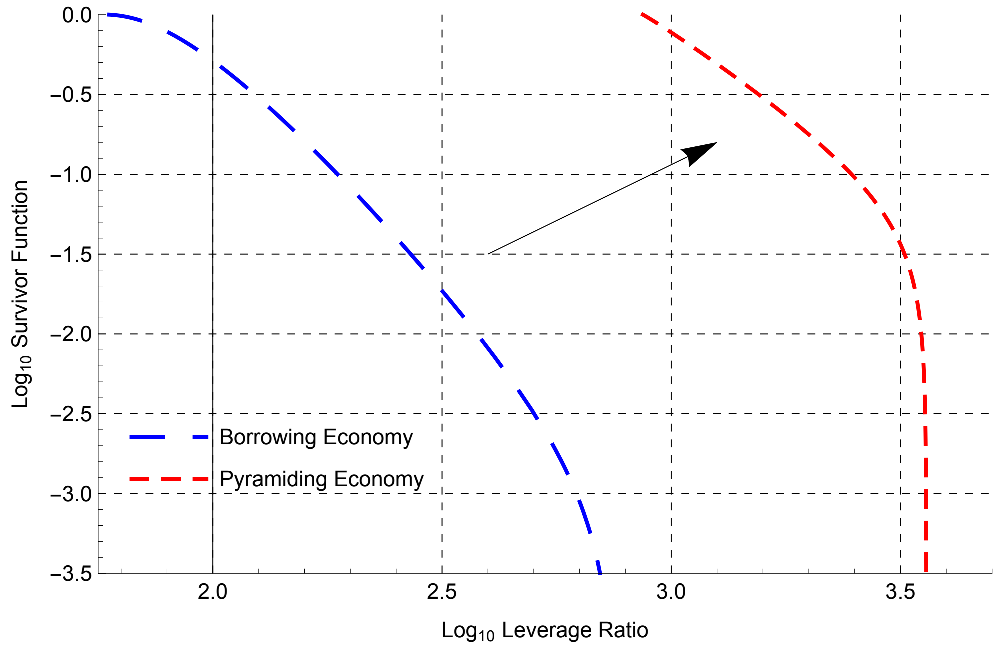
D.2 Pyramiding Economy

Figure 13: Investment Type, f = Uniform



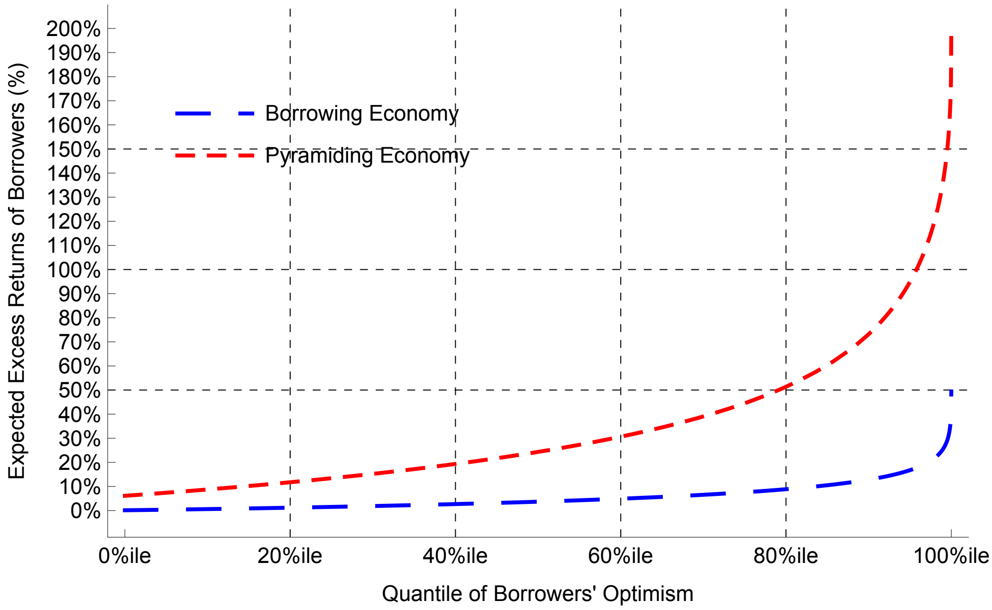
Note: With pyramiding, the price reflects even more the opinion of the most extreme optimists.

Figure 14: Leverage Ratio Distributions, Pareto Plot, f = Beta$(3,3)$, $\Delta =$ 2%



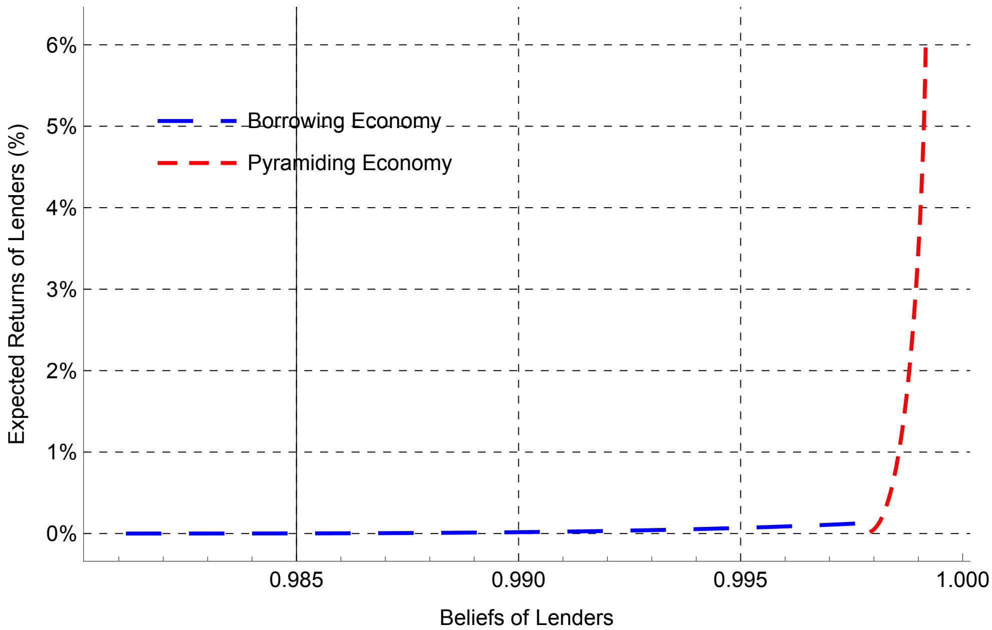
Note: Final borrowers' leverage result from the multiplication of two leverage ratio distributions in the pyramiding economy, with a lower Pareto tail coefficient. The empirical counterparts are shown in Figures 21 and 22.

Figure 15: **Expected Returns of Borrowers**, $f = \text{Beta}(3,3)$, $\Delta = 2\%$



Note: Final borrowers have higher expected returns with pyramiding than with borrowing only.

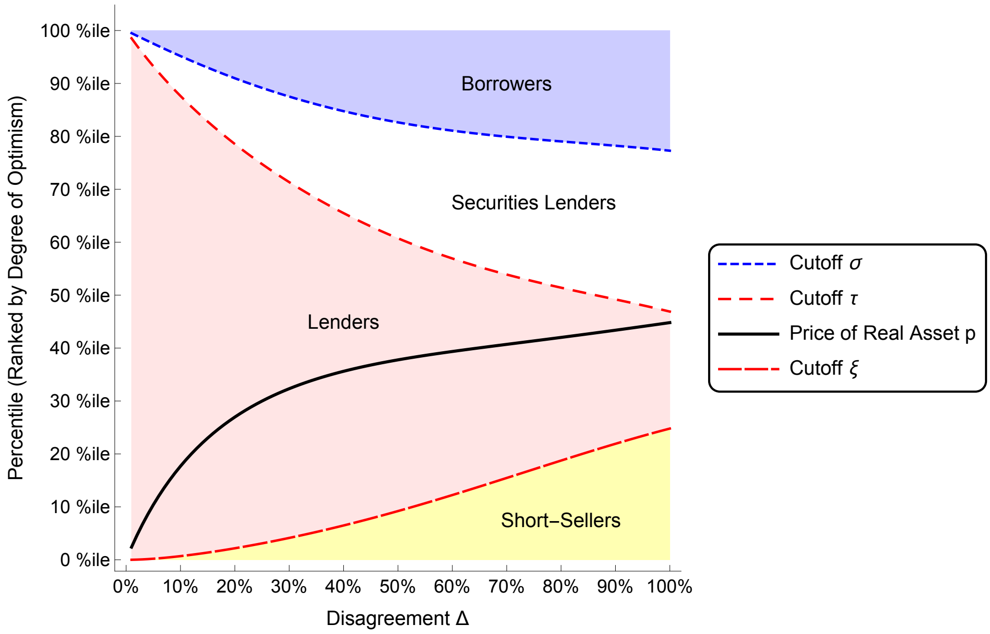
Figure 16: **Expected Returns of Lenders**, $f = \text{Beta}(3,3)$, $\Delta = 2\%$



Note: This Figure shows that direct lenders to borrowers ("Lenders" or "Lenders of Type 1") are endogenously more optimistic in Pyramiding Economy than in Borrowing Economy. It also shows that leveraging from lenders of type 2 leads them to considerably increase their expected returns.

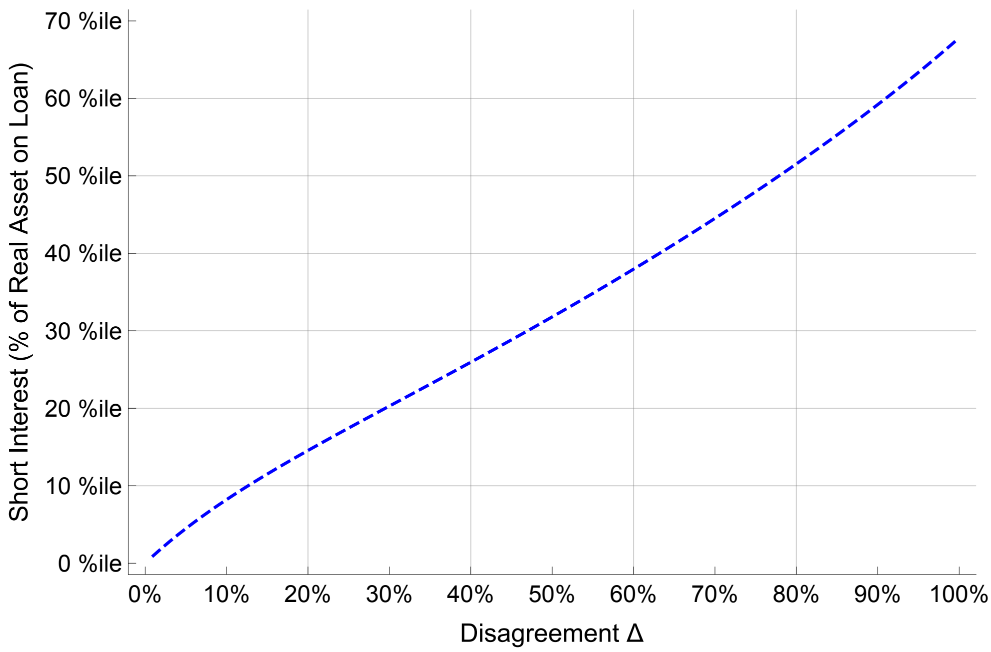
D.3 Borrowing with Short-Sales Economy

Figure 17: **Investment Type**, f = Uniform



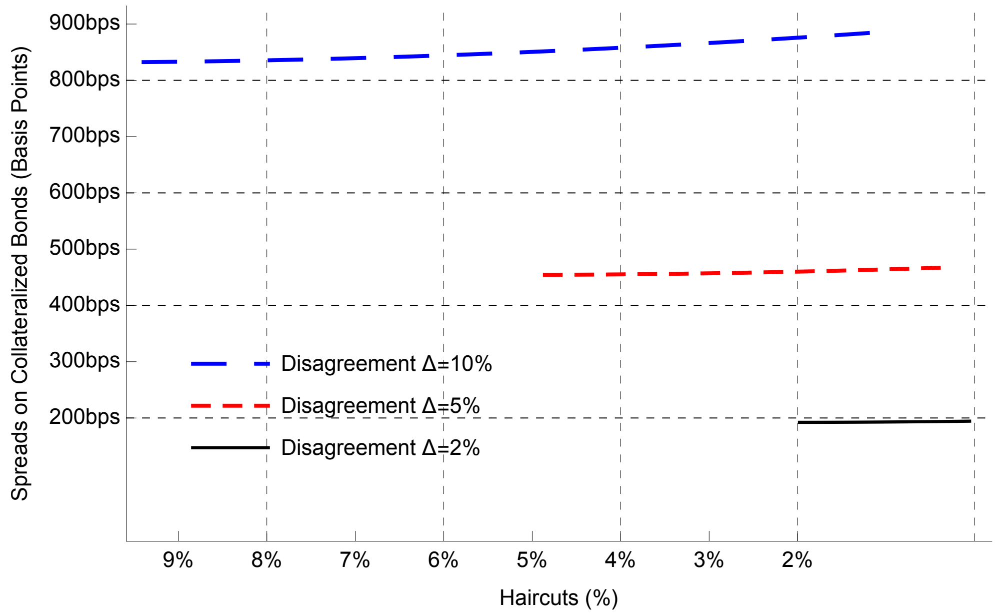
Note: All agents can both short and lend ex-ante, but some short while some lend. The price of the asset tends to represent that of the most pessimistic agents, as $\Delta \to 0$.

Figure 18: **Short Interest**, f = Uniform



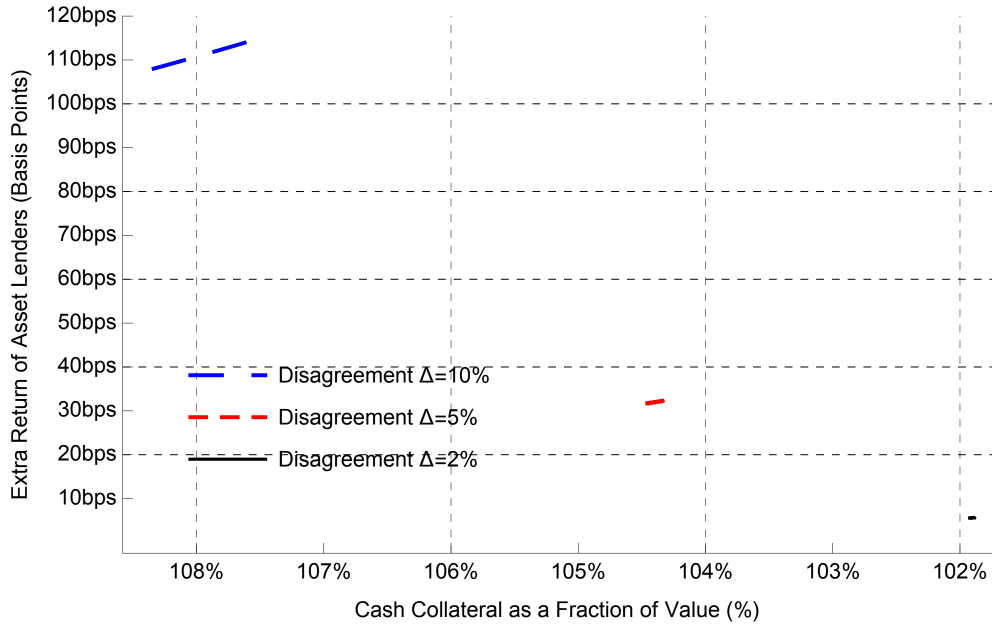
Note: Although all assets can potentially be loaned out, only a (rather small) fraction of them are in equilibrium. Shorting demand is limited as borrowers offer high rates to pessimists to lend cash.

Figure 19: **Hedonic Spreads** and Haircuts, f = Uniform



Note: On top of the assignment mechanism, returns on bonds are now higher than the risk free rate to incentivize lenders, who are natural short-sellers of the asset, to lend their wealth rather than use it as collateral to lever into short-selling bets.

Figure 20: **Rebate Rates** and Cash Collateral, f = Uniform

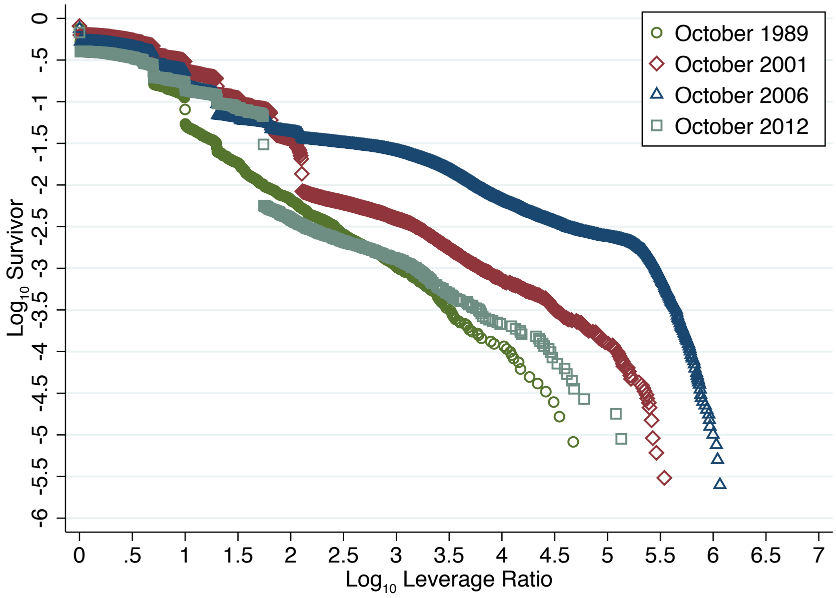


Note: Asset lenders earn an extra return, because they do not rebate the whole return to cash to short-sellers. This is a cost to short-sellers. These costs come from assignment, but also because asset lenders cannot leverage into the asset.

E Evidence

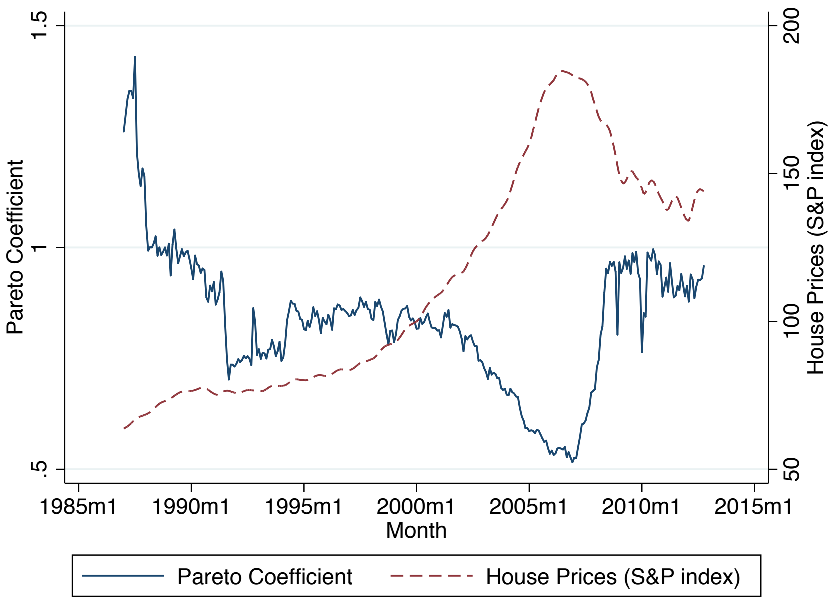
E.1 Housing

Figure 21: Leverage Ratio Distribution of US Homeowners (Leverage Ratio on New Loans) - Monthly Cross-Section



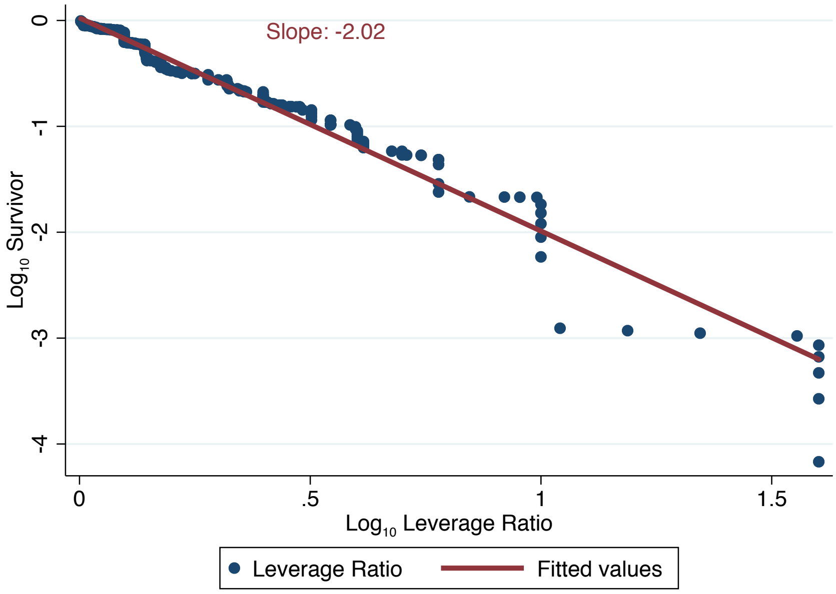
Note: In an online video, I show the universe of the cross-sections of leverage ratios between 1989 and 2012, aggregated monthly. This aggregates between 200000 and 500000 data points every month. Figure 14 shows that pyramiding leads to such an evolution of the leverage ratio distribution.

Figure 22: US House Prices and Pareto Coefficient of the Leverage Ratio Distribution



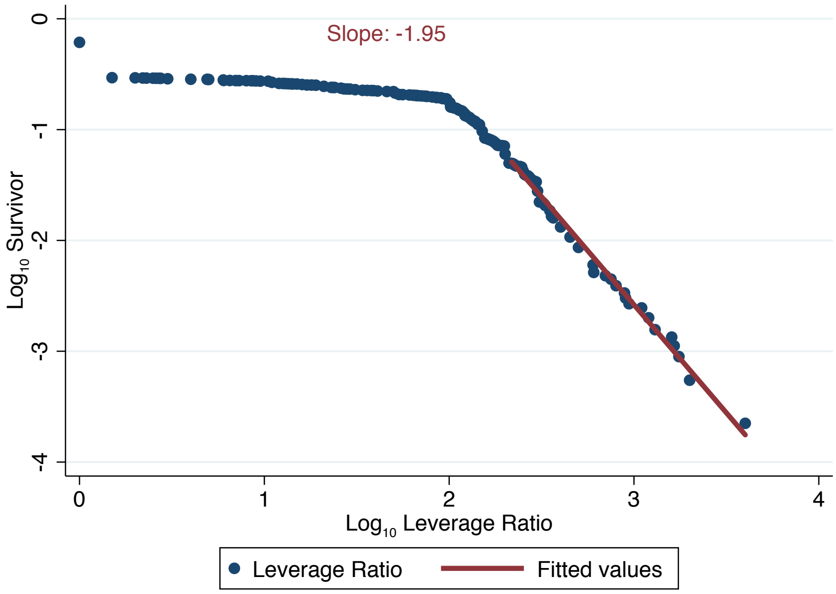
E.2 Entrepreneurs and Hedge Funds

Figure 23: Initial Leverage Ratio Distribution for Entrepreneurs (Source: SCF 2013)



Note: This Figure shows the initial leverage ratio distribution for entrepreneurs in the Survey of Consumer Finances, in year 2013.

Figure 24: Distribution of Hedge Funds' Leverage Ratios.



Note: This Figure shows the distribution of leverage ratio distribution for Hedge Funds, from the TASS Lipper Database in August 2006, comprising about 50% of the universe of hedge funds. (the point estimate of the Pareto tail coefficient is -1.95, with a standard error of 0.2, and an $R^2 = 98\%$).